# The Blum Medial Linking Structure for Multi–Region Analysis

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# Abstract

ELLEN GASPAROVIC: The Blum Medial Linking Structure for Multi–Region Analysis
(Under the direction of James Damon)

The Blum medial axis of a region with smooth boundary in $\mathbb{R}^{n+1}$ is a skeleton-like topological structure that captures shape and geometric properties of the region and its boundary. We introduce a structure, called the Blum medial linking structure, which extends the advantages of the medial axis to configurations of multiple disjoint regions in order to capture both their individual and "positional" or relative geometry. We use singularity theory to classify the generic local normal forms of the medial linking structure for generic configurations of regions in dimensions $n \leq 6$, which requires proving a transversality theorem for families of "multi–distance functions." We show how invariants of the geometry of the regions and their complement may be computed directly from the linking structure. We conclude with applications of the linking structure to the analysis of multiple objects in medical images.

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# Introduction

We consider a collection of objects in $\mathbb{R}^{n+1}$ modeled by disjoint regions with smooth boundaries. For example, such a collection of regions in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be used to model a collection of objects in a 2D or 3D medical image, which may consist of organs, glands, arteries, bones, etc. The analysis of one object in the image can be improved using knowledge of its geometrical relation to neighboring objects (see, e.g., [**24**] [**30**] [**41**]). Our objective is to introduce a structure which will capture both the geometric and shape properties of the individual objects as well as their "positional geometry," which captures the geometry of the regions relative to one another.

In the case of a single region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary $\mathcal{B}$, there have been a number of approaches to capturing the shape of $\Omega$. These include the *chordal locus* of Brady and Asada [**5**], the *arc–segment medial axis* of Leyton [**28**], the *symmetry set* of Bruce, Giblin, and Gibson [**8**], and the *Blum medial axis* [**2**] [**3**]. Each of these is a skeletal–like structure in $\Omega$ which captures shape properties. However, the Blum medial axis has proven to be the most useful, and it is the structure we will be extending to a configuration of regions. The concept of the medial axis was introduced by Blum, an engineer, in 1967 [**2**] [**3**], and was later independently introduced by Milman as the *central set* [**38**]. The *Blum medial axis* $M$ of $\Omega$ is the locus of centers of spheres in $\Omega$ which are tangent to $\mathcal{B}$ at two or more points (or have a degenerate tangency). On $M$ is defined a multivalued vector field $U$ from points of $M$ to the points of tangency.

The strengths of the Blum medial axis arise from three directions. First, for generic regions, the local structure of $M$ as a stratified set has been classified using singularity theory by Yomdin ($n \leq 3$) [**47**] and Mather ($n \leq 6$) [**32**] (with more detailed analysis for $n = 1, 2$ by Giblin and Kimia [**20**]). Second, $(M, U)$ belongs to a more general class of "skeletal structures" introduced by Damon, who has shown for such structures that the "medial geometry" of the radial vector field on $M$ completely captures the local and global geometry of the boundary and the region [**13**] [**14**] [**15**]. Third, a discrete version of the Blum medial axis and radial vector field, known as a discrete "m–rep" (see [**42**] and the many references in [**43**]), has been an especially effective tool for problems in medical imaging.

Our goal in introducing the Blum medial linking structure is to extend these advantages to configurations of multiple regions. The linking structure provides a means of relating the individual medial axes to one another and to the medial axis of the complementary region. First, we use singularity theory to classify the local normal forms of the medial linking structure for generic configurations of regions in dimensions $n \leq 6$. The genericity results in the multi–region setting require proving a transversality theorem for families of "multi–distance functions" defined on a collection of hypersurfaces, which extends that of Looijenga [**29**] for a single distance function used in Mather's proof. This will require that multiple transversality statements for different distance functions be satisfied at the same points. It is this simultaneity that brings in additional subtleties not present in the case of a single distance function.

Second, we extend the medial geometry of a single region to the entire configuration of regions including their complement. This leads into the third goal of applying the linking structure to address questions from mathematics and medical imaging for multi–region shape analysis; these questions are enumerated in Section 2.6 of Chapter 2. We identify invariants of the positional geometry of a configuration of regions, which may be computed directly from the linking structure, and construct a "tiered graph" structure involving the invariants that measures order of importance among regions.

In Chapter 1, we recall the details of the classification of the local structure of the Blum medial axis of a generic region in $\mathbb{R}^{n+1}$ for $n \leq 6$. We present a survey of the results for the medial geometry of a single region in Chapter 2. In Chapter 3, we state the generic linking classification theorems for dimensions $n = 1, 2$ as well as theorems on refinements of the stratifications of the boundaries and medial axes to reflect interactions between regions. We also define multivalued linking vector fields and prove their generic properties. In Chapter 4, we state and prove the transversality theorem for multi–distance functions, which is Theorem 4.3.1. We apply the transversality theorem in Chapter 5 to a collection of submanifolds and stratified sets in jet space in order to prove the collection of theorems stated in Chapter 3. Then, in Chapter 6, we define a piecewise smooth linking flow on each medial axis within a linking structure, and show how it can be used to capture both the geometry of individual regions and the geometry in the complement. Finally, we introduce several measures of comparison of a collection of regions and organize the results into a tiered graph structure.

CHAPTER 1

# Determining the Generic Structure of the Blum Medial Axis Using Singularity Theory

## 1.1. Introduction

The objective of this chapter is to present the details of the classification of the generic structure of the Blum medial axis of a single region. The first half of the chapter focuses on many of the fundamental definitions and theorems from singularity theory as developed by Thom [**44**] and Mather [**32**] (see also [**10**], [**31**], [**36**]). In Section 1.2, we recall the notion of right–equivalence of function germs, as well as the notion of a versal unfolding of such a germ and the infinitesimal criterion for versality. Then, in Section 1.3, we examine the relationship between versality and transversality.

In the second half of the chapter, we introduce the Blum medial axis as the Maxwell set of the family of distance squared functions associated to a given hypersurface in $\mathbb{R}^{n+1}$. In Section 1.4, we present a transversality theorem due to Looijenga [**29**], which Mather [**32**] applied to completely classify for dimensions $n \leq 6$ the generic structure of the Maxwell set. (Yomdin [**47**] gave a classification for $n \leq 3$ using a different method.) We define Whitney stratified sets in Section 1.5 and give a specific description of the stratification we obtain in $\mathbb{R}^3$. In Section 1.6, we finish with some remarks on the various notions of codimension that we shall use in this thesis.

## 1.2. $\mathcal{R}^+-$equivalence and versal unfoldings

In this section, we recall the notions of right equivalence of germs of smooth functions and versal unfoldings of such germs.

Let $X$ be a smooth $n$–dimensional manifold and let $x \in X$. Consider the smooth functions $f : U_1 \to \mathbb{R}$ and $g : U_2 \to \mathbb{R}$ where $U_1$ and $U_2$ are open neighborhoods of $x$ in $X$. We say that $f, g$ are *locally equivalent at* $x$ if there exists a neighborhood $x \in U \subset U_1 \cap U_2$ such that $f|U = g|U$. This defines an equivalence relation. A local equivalence class at $x$ is called a *germ* of a smooth function at $(X, x)$. We denote the germ determined by the function $f$ by $f : (X, x) \to (\mathbb{R}, y)$ with $f(x) = y$.

We can extend this notation to a finite set $S = \{x_1, \ldots, x_r\}$ of $r$ distinct points in $X$. Two smooth functions $f : U_1 \to \mathbb{R}$ and $g : U_2 \to \mathbb{R}$ with $U_1, U_2$ open and $S \subset U_1 \cap U_2$ are *locally equivalent at* $S$ if there is an open $U$, $S \subset U \subset U_1 \cap U_2$, so that $f|U = g|U$. This again forms an equivalence relation, and an equivalence class, denoted $f : (X, S) \to (\mathbb{R}, y)$ with $f(x_i) = y$ for $i = 1, \ldots, r$, is called a *multigerm* of a smooth function at $S$.

By choosing local coordinates at $x \in X$, we may now reduce to the case $X = \mathbb{R}^n$. The set of germs at $x$ of smooth functions $(\mathbb{R}^n, x) \to \mathbb{R}$ form a local ring which we denote by $\mathcal{E}_x$, with maximal ideal $\mathfrak{m}_x$ consisting of germs $(\mathbb{R}^n, x) \to (\mathbb{R}, 0)$. In the special case that $x = 0$, we let $\mathcal{E}_n$ denote the ring of germs at the origin and $\mathfrak{m}_n$ its unique maximal ideal. If $\boldsymbol{x} = (x^{(1)}, \ldots, x^{(n)})$ denote coordinates on $\mathbb{R}^n$, Hadamard's Lemma implies that $x^{(1)}, \ldots, x^{(n)}$ (viewed as function germs) generate the maximal ideal $\mathfrak{m}_n$ [**7**]. Similarly, let $\mathcal{E}_S$ denote the ring of multigerms of smooth functions at $S = \{x_1, \ldots, x_r\}$, so that if $\mathcal{E}_{x_i}$ denotes the ring of germs of smooth functions at $x_i \in \mathbb{R}^n$ for $i = 1, \ldots, r$,

$$\mathcal{E}_S = \mathcal{E}_{x_1} \oplus \ldots \oplus \mathcal{E}_{x_r}.$$

Also, let $\mathfrak{m}_S = \mathfrak{m}_{x_1} \oplus \ldots \oplus \mathfrak{m}_{x_r}$. Thus, the multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}, y)$ may be viewed as the $r$-tuple $(f_1, \ldots, f_r)$, where for each $i$, $f_i : (\mathbb{R}^n, x_i) \to (\mathbb{R}, y)$ is a germ of a smooth function at $x_i \in \mathbb{R}^n$.

We next recall the notion of *right–equivalence* (see, e.g., [**33**], [**32**]).

**Definition 1.2.1.** *(a) Let $f, g \in \mathcal{E}_x$. We say $f$ is **right–equivalent**, or $\mathcal{R}$–**equivalent**, to $g$ if there exists a germ of a diffeomorphism $\phi : (\mathbb{R}^n, x) \to (\mathbb{R}^n, x)$ satisfying*

$$f = g \circ \phi.$$

*We say $f$ is $\mathcal{R}^+$–**equivalent** to $g$ if, for some constant $c \in \mathbb{R}$ and $\phi$ as above,*

$$f = g \circ \phi + c.$$

*(b) Two smooth multigerms $f, g \in \mathcal{E}_S$ are said to be $\mathcal{R}$–**equivalent** if there exists a multigerm of a smooth diffeomorphism $\phi : (\mathbb{R}^n, S) \to (\mathbb{R}^n, S)$, with $\phi(x_i) = x_i$ for $1 \leq i \leq r$, such that*

$$f = g \circ \phi.$$

*With such a $\phi$, $f$ and $g$ are said to be $\mathcal{R}^+$–**equivalent** if*

$$f = g \circ \phi + c$$

*for some $c \in \mathbb{R}$. That is, letting $f = (f_1, \dots, f_r)$, $g = (g_1, \dots, g_r)$, and $\phi = (\phi_1, \dots, \phi_r)$, the statement that $f$ is $\mathcal{R}^+$-equivalent to $g$ translates to*

$$(f_1, \dots, f_r) = (g_1 \circ \phi_1 + c, \dots, g_r \circ \phi_r + c).$$

Observe that $\mathcal{R}$–equivalence for multigerms requires that the value of $f$ and $g$ at the points of $S$ be the same, while $\mathcal{R}^+$–equivalence allows $f$ and $g$ to each take on a different value at $S$.

Next, we introduce the notion of an *unfolding* family of functions of an initial multigerm $f$.

**Definition 1.2.2.** *For $f \in \mathcal{E}_S$, the smooth map germ*

$$F : (\mathbb{R}^n \times \mathbb{R}^s, (S, 0)) \to \mathbb{R},$$

$$(x, w) \mapsto F_w(x) = F(x, w)$$

*with $F_0(x) = f(x)$ is said to be an $s$–**parameter unfolding** of $f$.*

Following [**10**], let $\mathcal{R}^+_{un}(s)$ denote the group of $s$–parameter unfoldings, which acts on $s$–parameter unfoldings of germs $f \in \mathcal{E}_S$.

**Definition 1.2.3.** *Let $F, G : (\mathbb{R}^n \times \mathbb{R}^s, (S, 0)) \to \mathbb{R}$ be unfoldings of $f \in \mathcal{E}_S$. The families $F, G$ are said to be $\mathcal{R}^+_{un}$–**equivalent** if there exists a multigerm of a diffeomorphism*

$$\Phi : (\mathbb{R}^n \times \mathbb{R}^s, (S, 0)) \to (\mathbb{R}^n \times \mathbb{R}^s, (S, 0)),$$
$$(x, w) \mapsto (\phi_1(x, w), \Psi(w)),$$

*and a smooth function germ $C : (\mathbb{R}^s, 0) \to \mathbb{R}$ such that*

$$F(x, w) = G(\phi_1(x, w), \Psi(w)) + C(w).$$

**Definition 1.2.4.** *Suppose $F$ is an $s$–parameter unfolding of $f \in \mathcal{E}_S$, while $G$ is a $p$–parameter unfolding of $f$. A **mapping from** $G$ **to** $F$ consists of a smooth germ $\psi : (\mathbb{R}^p, 0) \to (\mathbb{R}^s, 0)$ such that the unfolding*

$$\psi^* F : (\mathbb{R}^n \times \mathbb{R}^p, (S, 0)) \to \mathbb{R},$$
$$(x, w) \mapsto F(x, \psi(w))$$

*is $\mathcal{R}^+_{un}$–equivalent to $G$. The unfolding $\psi^* F$ is called the **pullback** of $F$ by $\psi$.*

We arrive at the definition of a *versal unfolding.*

**Definition 1.2.5.** *Let $F : (\mathbb{R}^n \times \mathbb{R}^s, (S, 0)) \to \mathbb{R}$ be an unfolding of the smooth multigerm $f \in \mathcal{E}_S$. Then $F$ is said to be **versal** if, for any other unfolding $G : (\mathbb{R}^n \times \mathbb{R}^p, (S, 0)) \to \mathbb{R}$ of $f$, there exists $\psi : (\mathbb{R}^p, 0) \to (\mathbb{R}^s, 0)$ such that $G$ and $\psi^* F$ are $\mathcal{R}^+_{un}$–equivalent.*

**1.2.1. Infinitesimal versality.** In this section, we give an infinitesimal criterion for an $s$–parameter unfolding $F$ of a germ $f \in \mathcal{E}_n$ or a multigerm $f \in \mathcal{E}_S$ to be versal.

Define in $\mathcal{E}_n$ the Jacobian ideal

$$J(f) = \langle \partial f \rangle = \mathcal{E}_n \cdot \left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\} \tag{1.1}$$

and the $\mathbb{R}$–vector subspace

$$\langle \partial F \rangle = \mathbb{R} \cdot \left\{ \left.\frac{\partial F}{\partial u_1}\right|_{u=0}, \ldots, \left.\frac{\partial F}{\partial u_s}\right|_{u=0} \right\}. \tag{1.2}$$

The *extended tangent space to the* $\mathcal{R}^+$*–orbit of* $f$ is

$$T\mathcal{R}_e^+ f = \langle \partial f \rangle + \langle 1 \rangle \tag{1.3}$$

and the *extended normal space to the* $\mathcal{R}^+$*–orbit of* $f$ is the space

$$N\mathcal{R}_e^+ \cdot f = \mathcal{E}_n / T\mathcal{R}_e^+ \cdot f. \tag{1.4}$$

(The usual tangent space is given by $T\mathcal{R}^+ f = \mathfrak{m}_n \cdot \left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\} + \langle 1 \rangle$, but the extended tangent space removes the restriction that 0 must map to 0.)

**Definition 1.2.6.** *Let* $f \in \mathcal{E}_n$*. The* $\mathcal{R}_e^+$*–**codimension of*** $f$ *is*

$$\mathcal{R}_e^+\text{–}\boldsymbol{codim}(f) = dim_{\mathbb{R}} \ \ N\mathcal{R}_e^+ \cdot f.$$

**Definition 1.2.7.** *Let* $F$ *be an unfolding of the germ* $f$ *at* $x \in \mathbb{R}^n$*. We say* $F$ *is an **infinitesimally*** $\mathcal{R}^+$***–versal unfolding*** *of* $f$ *if*

$$\mathcal{E}_n = T\mathcal{R}_e^+ \cdot f + \langle \partial F \rangle.$$

Next, we give the analogous definitions in the multigerm setting. Let $F$ be an $s$–parameter unfolding of $f = (f_1, \ldots, f_r) \in \mathcal{E}_S$, so that $F = (F_1, \ldots, F_r)$ where $F_i$ is an unfolding of $f_i$ for every $i = 1, \ldots, r$. Define in $\mathcal{E}_S$ the ideal

$$\langle \partial f \rangle = \langle \partial f_1 \rangle \times \ldots \times \langle \partial f_r \rangle$$

and the vector subspace

$$\langle \partial F \rangle = \langle \partial_1 F, \ldots, \partial_s F \rangle,$$

where $\partial_j F = (\partial_j F_1, \ldots, \partial_j F_r)$ for $j = 1, \ldots, s$ and $\partial_j F_i = \left.\dfrac{\partial F_i}{\partial u_j}\right|_{u=0}$ for every $i = 1, \ldots, r$. As in the single germ setting, the *extended tangent space to the* $\mathcal{R}^+$*−orbit of* $f$ is

$$T\mathcal{R}_e^+ f = \langle \partial f \rangle + \langle 1 \rangle, \tag{1.5}$$

the *extended normal space to the* $\mathcal{R}^+$*−orbit of* $f$ is the space

$$N\mathcal{R}_e^+ \cdot f = \mathcal{E}_S / T\mathcal{R}_e^+ \cdot f, \tag{1.6}$$

and the *extended* $\mathcal{R}^+$*−codimension of* $f$ is defined as in Definition 1.2.6.

**Definition 1.2.8.** *Let $F$ be an unfolding of $f \in \mathcal{E}_S$. Then $F$ is an* ***infinitesimally*** $\mathcal{R}^+$***−versal unfolding*** *of $f$ if*

$$\mathcal{E}_S = T\mathcal{R}_e^+ \cdot f + \langle \partial F \rangle.$$

For both single and multigerms, we have the following fundamental theorem which was stated for germs by Thom [**44**] and proven by Mather ([**33**]). The version for multigerms was stated by Mather in [**32**] and a proof follows from the general unfolding theorem in [**16**].

**Theorem 1.2.9.** *An unfolding is versal if and only if it is infinitesimally versal.*

Thus, the infinitesimal criterion for versality implies that the $\mathcal{R}_e^+$−codimension of $f$ gives the minimum number of parameters that are needed for $f$ to be versally unfolded. A versal unfolding $F$ is said to be *miniversal* if $F$ has the least possible number of parameters.

An important consequence of Theorem 1.2.9 is that it supplies a way to construct versal unfoldings. The following corollary is for a versal unfolding of a single germ, but the analogous result holds for a versal unfolding of a multigerm.

**Corollary 1.2.10.** *Let $f \in \mathcal{E}_n$. Suppose the $\mathcal{R}_e^+-$codimension of $f$ is $s$, and let $w_1, \ldots, w_s$ be a basis for $N\mathcal{R}_e^+ \cdot f$. Then*

$$F(x,u) = f(x) + u_1 w_1(x) + \ldots + u_s w_s(x)$$

*is a miniversal unfolding of $f$.*

Mather proved the following essential theorem on the uniqueness of versal unfoldings, which follows from Theorem 1.2.9.

**Theorem 1.2.11** (Mather [**33**]). *Any two $s$–parameter $\mathcal{R}^+-$miniversal unfoldings are isomorphic.*

Therefore, any miniversal unfolding has an algebraic normal form. Moreover, any $p$–parameter versal unfolding $G$ with $p > s$ is isomorphic to the unfolding $F \times \mathrm{id}_{p-s}$ with $F$ as in Corollary 1.2.10, so $G$ will also have a polynomial normal form.

In addition to the uniqueness theorem, there is another important theorem regarding versality which establishes its openness property.

**Theorem 1.2.12.** *Let $F : (\mathbb{R}^n \times \mathbb{R}^s, (S, 0)) \to \mathbb{R}$ be an $\mathcal{R}^+-$versal unfolding of the multigerm $f \in \mathcal{E}_S$, and let*

$$\Sigma(F) = \left\{ (x,w) \in \mathbb{R}^n \times \mathbb{R}^s : \frac{\partial F}{\partial x^{(i)}}(x,w) = 0,\, i = 1, \ldots, n \right\}$$

*be the critical set of the unfolding. Then there exist neighborhoods $W$ of $0$ in $\mathbb{R}^s$ and $U$ of $S$ in $\mathbb{R}^n$ so that, for all $w \in W$ and $S' \subset \Sigma(F) \cap (U \times \{w\})$, $F : (\mathbb{R}^n \times \mathbb{R}^s, (S', w)) \to \mathbb{R}$ is an $\mathcal{R}^+-$versal unfolding.*

Finally, we introduce the notion of finite determinacy of germs [**36**], which requires the notion of a *$k$–jet* of a smooth function. If $f : (\mathbb{R}^n, x) \to \mathbb{R}$ is a smooth germ, the $k$–jet of $f$ at $x$, denoted $j^k f(x)$, is the list of derivatives

$$j^k f(x) = \left( f(x), df(x), d^2 f(x), \ldots, d^k f(x) \right),$$

which is equivalent to giving the $k$–th order Taylor expansion of $f$ at $x$.

**Definition 1.2.13.** *Let $f \in \mathcal{E}_n$ and suppose that, for some positive integer $k$, any $g \in \mathcal{E}_n$ with $j^k f(0) = j^k g(0)$ is $\mathcal{R}^+-$equivalent to $f$. Then $f$ is said to be* ***finitely $k-$determined for $\mathcal{R}^+-$equivalence****.*

Finite determinacy of $f$ implies that it is sufficient to study the $k$–jet of a finitely determined germ to determine its singularity theoretic properties. If $f, g \in \mathcal{E}_n$ (or $\mathcal{E}_S$) are $\mathcal{R}^+-$equivalent germs, then $f$ is $k$–determined if and only if $g$ is [**36**]. We conclude with a final theorem involving finite determinacy.

**Theorem 1.2.14** (Mather [**36**])**.** *A germ $f \in \mathcal{E}_n$ (or $\mathcal{E}_S$) is finitely determined if and only if $f$ is of finite $\mathcal{R}^+-$codimension.*

Consequently, by the infinitesimal criterion for versality, a germ $f$ has an $\mathbb{R}^+$–versal unfolding if and only if $f$ is finitely determined.

In Section 1.3, we shall examine more consequences and fundamental theorems of versality.

## 1.3. Versality and transversality

There is a fundamental relationship between versality of unfoldings and transversality to certain submanifolds of jet space. Before delving into this relationship, we briefly recall some notation related to jet spaces (see, e.g., [**23**]).

For smooth manifolds $X^n$ and $Y^p$, let $J^k(X, Y)$ denote the $k$–jet bundle of maps $X \to Y$ with base $X$ and fiber $J^k(\mathbb{R}^n, \mathbb{R}^p)_x$, the space of jets of functions $\mathbb{R}^n \to \mathbb{R}^p$ at the point $x \in \mathbb{R}^n$. Alternatively, $J^k(X, Y)$ may be viewed as a bundle over $X \times Y$ with fiber consisting of $k$–jets of mappings $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, which we denote by $J^k(n, p)$. If $f : X \to Y$ is smooth,

$$j^k f : X \to J^k(X, Y)$$

$$x \mapsto j^k f(x)$$

denotes the $k$–jet extension mapping, which is also smooth. Let $\alpha : J^k(X,Y) \to X$ and $\beta : J^k(X,Y) \to Y$ denote the source and target mappings, respectively. Next, let $X^{(r)} = \underbrace{X \times \ldots \times X}_{r \text{ times}} \setminus \Delta X$, where $\Delta X$ is the generalized diagonal in $X^r$, i.e.,

$$\Delta X = \{(x_1, \ldots, x_r) \in X^r : x_i = x_j \text{ for some } i \neq j\}.$$

The $r$–multi $k$–jet bundle is ${}_rJ^k(X,Y)$ with fiber ${}_rJ^k(\mathbb{R}^n, \mathbb{R}^p)_S$ for $S = \{x_1, \ldots, x_r\}$ and base $X^{(r)}$, and the $r$–multi $k$–jet extension of $f : X \to Y$ is

$$\begin{aligned} &{}_rj^k f : X^{(r)} \to {}_rJ^k(X,Y) \\ &(x_1, \ldots, x_r) \mapsto {}_rj^k f(x_1, \ldots, x_r) = (j^k f(x_1), \ldots, j^k f(x_r)). \end{aligned}$$

The group $\mathcal{R}^{(k)}$ of $k$-jets of diffeomorphism germs $\mathbb{R}^n \to \mathbb{R}^n$ acts algebraically on the jet space fiber $J^k(\mathbb{R}^n, \mathbb{R})_x$. Namely, if $j^k\phi \in \mathcal{R}^{(k)}$, $z \in J^k(\mathbb{R}^n, \mathbb{R})$ with $f \in \mathcal{E}_n$ such that $j^k f$ is a representative of $z$, and $c \in \mathbb{R}$, then the action is given by

$$j^k\phi \cdot z = j^k(f \circ \phi) + c. \tag{1.7}$$

We obtain a natural action on the multijet space fiber ${}_rJ^k(\mathbb{R}^n, \mathbb{R})_S$ in the analogous way, i.e., if $\phi = (\phi_1, \ldots, \phi_r)$ is a multigerm of a diffeomorphism at $S$, $z = (z_1, \ldots, z_r) \in {}_rJ^k(\mathbb{R}^n, \mathbb{R})_S$ with $j^k f_i$ a representative of $z_i$ for $i = 1, \ldots, r$, and $c \in \mathbb{R}$, we have

$$j^k\phi \cdot z = (j^k(f_1 \circ \phi_1) + c, \ldots, j^k(f_r \circ \phi_r) + c). \tag{1.8}$$

In this case, we denote the group of $k$–jets of diffeomorphism multigerms by ${}_r\mathcal{R}^+$. Because the action of ${}_r\mathcal{R}^+$ on multijets is that of an algebraic group, it follows that these multi–orbits in ${}_rJ^k(\mathbb{R}^n, \mathbb{R})_S$ under the group action are submanifolds (see Mather [**37**]). The *simple singularities*, which consist of the families of $A$, $D$, and $E$ singularities (see, e.g., [**1**]), are those orbits for which there are only a finite number of other orbits within a sufficiently small neighborhood. We now recall the definitions of $A_k$ and $\mathcal{A}_\beta$ *singularities.*

**Definition 1.3.1.** *(a) The smooth germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *is said to have an* $A_k$ ***singularity at 0*** *for some positive integer* $k$ *if* $f$ *is* $\mathcal{R}^+-$*equivalent to the smooth germ*

$$g = \sum_{i=1}^{n-1} \pm x_i^2 \pm x_n^{k+1}.$$

*(b) The smooth multigerm* $f : (\mathbb{R}^n, S) \to (\mathbb{R}, 0)$ *with* $S = \{x_1, \ldots, x_r\}$ *is said to have an* $\mathcal{A}_\beta$ ***singularity at*** $S$ *for an* $r-$*tuple* $\beta = \{k_1, \ldots, k_r\}$ *if* $f$ *is* $\mathcal{R}^+-$*equivalent to*

$$g = \left( \sum_{i=1}^{n-1} (x_1^{(i)})^2 + (x_1^{(n)})^{k_1+1}, \ldots, \sum_{i=1}^{n-1} (x_r^{(i)})^2 + (x_r^{(n)})^{k_r+1} \right),$$

*where* $(x_i^{(1)}, \ldots, x_i^{(n)})$ *are coordinates on the* $i$*-th copy of* $\mathbb{R}^n$ *for* $i = 1, \ldots, r$*.*

These are the only simple multigerms with each germ having a local minimum at 0. In part (b) above, source coordinates may be chosen independently around each $x_i \in S$ so that $x_i$ is locally the origin in each copy of $\mathbb{R}^n$. As we shall see in the next section, the $A_k$ and $\mathcal{A}_\beta$ singularities are the relevant orbits for the classification of the generic structure of the medial axis.

**Remark 1.3.2.** Other examples of submanifolds of jet space include the *Thom-Boardman singularity submanifolds.* For a smooth map $f : X \to Y$ with $X^n$ and $Y^p$ smooth manifolds, we may decompose the source space $X$ by singularity type of $f$ in the following way. Let corank$(f) = \min(n, p) - (\text{rank}(f) \text{ at } x)$, and define

$$S^i(f) = \{x \in X : \textit{corank}(f) = i\}.$$

If $f$ has corank 0, the $k$-jet of $f$ is said to be regular. Thom proved that the sets $S^i(f)$ are submanifolds of $X$ [**27**]. Similarly, we may define the sets

$$S^{i,j}(f) = S^j(f|S^i(f))$$

for some nonnegative integer $j$; continuing the process inductively, we define, for the sequence $I = \{i_1, ..., i_j\}$ with integers $i_1 \geq \ldots \geq i_j \geq 0$, the sets

$$S^I(f) = S^j(f|S^{i_1,...,i_{j-1}}(f)).$$

In [**4**], Boardman proved that a set of the form $S^I(f)$ is a manifold by finding a submanifold $\Sigma^I(X,Y)$ of $J^k(X,Y)$ such that the jet extension mapping

$$j^k f : X \to J^k(X,Y)$$

satisfies $j^k f \mathrel{\overline{\pitchfork}} \Sigma^I(X,Y)$ and $j^k f^{-1}(\Sigma^I(X,Y)) = S^I(f)$. We refer to the submanifold $\Sigma^I(X,Y)$ as the *Boardman manifold with symbol $I$*. Let $\Sigma^I$ denote the fiber of $\Sigma^I(X,Y)$, and let $\Sigma^{1_j} = \Sigma^{1,...,1}$ with 1 appearing $j$ times.

Now, let $F$ be an $s$–parameter unfolding of $f \in \mathcal{E}_n$. The $k$–jet extension of $F$ is the mapping

$$j_1^k F : \mathbb{R}^n \times \mathbb{R}^s \to J^k(\mathbb{R}^n, \mathbb{R}),$$
$$(x,u) \mapsto j_1^k F(x,u) = j^k F(\cdot, u)(x),$$

so that the subscript "1" indicates the jet is taken with respect to the coordinates on $\mathbb{R}^n$. There is a natural algebraic identification of this jet space as

$$J^k(\mathbb{R}^n, \mathbb{R})_x \cong \mathcal{E}_n / \mathfrak{m}_n^{k+1} \tag{1.9}$$

(see, e.g., [**37**]), and the fiber $J^k(n,1)$ is identified with

$$J^k(n,1) \cong \mathfrak{m}_n / \mathfrak{m}_n^{k+1}. \tag{1.10}$$

Likewise, suppose $F$ denotes an $s$–parameter unfolding of the multigerm $f \in \mathcal{E}_S$. The $r$–multi $k$–jet extension of $F$ is the mapping

$$
\begin{aligned}
&{}_r j_1^k F : (\mathbb{R}^n)^{(r)} \times \mathbb{R}^s \to {}_r J^k(\mathbb{R}^n, \mathbb{R}), \\
&\qquad (x_1, \ldots, x_r, u) \mapsto {}_r j_1^k F(x, u) = (j^k F(\cdot, u)(x_1), \ldots, j^k F(\cdot, u)(x_r)).
\end{aligned}
$$

We come now to the important theorem that enables one to express the condition that $F$ be a versal unfolding of $f \in \mathcal{E}_S$ as a transversality statement.

**Theorem 1.3.3** ([**32**]). *Let $F$ be an $s$-parameter unfolding of $f \in \mathcal{E}_S$. Then $F$ is an $\mathcal{R}^+-$versal unfolding of $f$ if and only if ${}_r j_1^k F$ is transverse to the orbit of ${}_r j_1^k F(S, 0)$ in ${}_r J^k(\mathbb{R}^n, \mathbb{R})$ under the action of ${}_r \mathcal{R}^+$ for sufficiently large $k$.*

## 1.4. The generic structure of Blum medial axis for $n \leq 6$

In this section, we present a variant of Thom's transversality theorem due to Looijenga, as well as an extension of it due to Wall, which apply to families of distance squared functions. Looijenga's theorem yields a complete classification of the generic structure of the Blum medial axis in dimensions $n \leq 6$. The genericity results hold for a residual set of embeddings, and are proven by demonstrating transversality of a jet extension of a mapping to certain submanifolds of a jet space, then applying the transversality theorem.

**1.4.1. Looijenga's transversality theorem and distance–genericity.** Suppose $X$ is a smooth, compact, connected, $n$-dimensional manifold, and $\phi : X \hookrightarrow \mathbb{R}^{n+1}$ a smooth embedding. Let

$$
\begin{aligned}
&\sigma_\phi : X \times \mathbb{R}^{n+1} \to \mathbb{R}, \\
&\qquad (x, w) \mapsto ||\phi(x) - w||^2
\end{aligned}
\tag{1.11}
$$

be the family of squared distance functions, and let

$$
\begin{aligned}
{}_r j_1^k \sigma_\phi : X^{(r)} \times \mathbb{R}^{n+1} \times V &\to {}_r J^k(X, \mathbb{R}) \\
(x_1, \ldots, x_r, w, v) &\mapsto (j_1^k(||\Phi(\cdot, v) - w||^2(x_1), \ldots, j_1^k(||\Phi(\cdot, v) - w||^2(x_r))
\end{aligned}
$$

denote the $r$–multi $k$–jet extension of $\sigma_\phi$. Looijenga's transversality theorem applies to a certain class of submanifolds $W$ of the multijet space ${}_r J^k(X, \mathbb{R})$: those $W$ which are *invariant under addition of constants*. This means that, for any $c \in \mathbb{R}$, $z = (z_1, \ldots, z_r) \in W$ implies that $(z_1 + c, \ldots, z_r + c) \in W$.

**Theorem 1.4.1** (Looijenga [**29**], [**45**]). *Let $W$ be a smooth submanifold of ${}_r J^k(X, \mathbb{R})$ that is invariant under addition of constants. Then*

$$\mathcal{H} = \{\phi \in \mathrm{Emb}(X, \mathbb{R}^{n+1}) : {}_r j_1^k \sigma_\phi \textit{ is transverse to } W\}$$

*is a residual set in the* $\mathrm{C}^\infty$ *topology. If $X$ is compact, the set is open and dense.*

We refer to the elements of $\mathcal{H}$ as *distance-generic embeddings*. In [**45**], Wall proved an extension of Looijenga's theorem. Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit $n$–sphere, let $X$ and $\phi$ be as in the statement of Theorem 1.4.1, and define the family of height functions

$$
\begin{aligned}
h_\phi : X \times S^n &\to \mathbb{R}, \\
(x, w) &\mapsto w \cdot \phi(x).
\end{aligned}
\tag{1.12}
$$

**Theorem 1.4.2.** *Let $W$ be a smooth submanifold of ${}_r J^k(X, \mathbb{R})$ that is invariant under addition of constants. Then*

$$\mathcal{H} = \{\phi \in \mathrm{Emb}(X, \mathbb{R}^{n+1}) : {}_r j_1^k h_\phi \textit{ is transverse to } W\}$$

*is a residual set in the* $\mathrm{C}^\infty$ *topology. If $X$ is compact, the set is open and dense.*

**1.4.2. Mather's classification of local normal forms of $M$.** In this section, we first recall how the Blum medial axis may be defined as the Maxwell set of the family $\sigma_\phi$.

Then, we explain how Looijenga's transversality theorem allows its generic structure to be determined.

**Definition 1.4.3.** *Given a smooth manifold $N$ and a smooth family $F : N \times \mathbb{R}^s \to \mathbb{R}$, the **Maxwell set of** $F$ is the set of parameters $w \in \mathbb{R}^s$ such that $F(\cdot, w)$ attains an absolute minimum value at more than one distinct point in $N$ or at a degenerate critical point.*

Let $\phi : X \hookrightarrow \mathbb{R}^{n+1}$ be a smooth embedding of a smooth compact, connected, $n$-dimensional manifold $X$, where $\phi(X) = \mathcal{B}$ is the boundary of a smooth, compact, connected region $\Omega \subset \mathbb{R}^{n+1}$ by the Jordan–Brouwer Separation Theorem. Then the medial axis of $\Omega$ is the part of the Maxwell set of $\sigma_\phi$ (defined in (1.11)) that lies in $\Omega$.

Mather and Yomdin classified the generic local normal forms of the medial axis, Yomdin for dimension $n \leq 3$ [**47**] and Mather for $n \leq 6$ [**32**]. Moduli appear in dimension 7 and higher, preventing further smooth classification (although a classification by topological equivalence becomes possible). Of the simple singularities, only the $A_{2k+1}$ singularities as defined in Definition 1.3.1 are relevant to the classification as they are the only simple singularities that have local minima.

**Theorem 1.4.4** (Mather [**32**], Yomdin [**47**])**.** *For $n \leq 6$, let $X$ be a smooth, compact, connected, $n$-dimensional manifold, and let $\phi : X \hookrightarrow \mathbb{R}^{n+1}$ be a smooth embedding with $\phi(X) = \mathcal{B}$, where $\mathcal{B} = \partial\Omega$. Locally, the Maxwell set of $\sigma_\phi$ is diffeomorphic to the Maxwell set of the $\mathcal{R}^+-$versal unfolding of one of the following germs:*

- $(1 \leq n \leq 6)$ $A_1^2$, $A_1^3$, $A_3$,
- $(2 \leq n \leq 6)$ $A_1^4$, $A_1A_3$,
- $(3 \leq n \leq 6)$ $A_1^5$, $A_5$, $A_1^2A_3$,
- $(4 \leq n \leq 6)$ $A_1^6$, $A_1A_5$, $A_3^2$, $A_1^3A_3$,
- $(n = 5, 6)$ $A_1^7$, $A_1^2A_5$, $A_1A_3^2$, $A_1^4A_3$, $A_7$,
- $(n = 6)$ $A_1^8$, $A_1A_7$, $A_1^3A_5$, $A_3A_5$, $A_1^2A_3^2$, $A_1^5A_3$.

**Remark 1.4.5.** Mather actually classified the generic normal forms for the family of distance functions, rather than distance squared. However, the normal forms are the same if one replaces the family of distance functions with the family of distance squared functions, which is smooth, due to the fact that these families are $\mathcal{R}^+-$equivalent as unfoldings and therefore have the same types of critical points.

## 1.5. Stratifications of the medial axis $M$ and boundary $\mathcal{B}$

In this section, we explain how to obtain stratifications of the boundary $\mathcal{B}$ and Blum medial axis $M$ of a region $\Omega \subset \mathbb{R}^{n+1}$ arising from the singular behavior related to the Maxwell set of the family of distance squared functions. We first recall the notion of a *Whitney stratification*, then give the specific details of the boundary and medial axis stratifications in $\mathbb{R}^3$.

**1.5.1. Whitney stratifications.** In this section, we recall the notation of a *stratified space* and, in particular, a *Whitney stratified space.*

**Definition 1.5.1.** *A closed set $M \subset \mathbb{R}^{n+1}$ is a* ***stratified set*** *if $M$ may be written as the union of a locally finite collection of smooth, locally closed, disjoint submanifolds $S_\alpha \subset \mathbb{R}^{n+1}$, $\alpha \in I$, where $I$ is a partially ordered index set. The submanifolds, called* ***strata****, must satisfy the axiom of the frontier: $S_\alpha \cap \overline{S_\beta} \neq \emptyset$ if and only if $S_\alpha \subset \overline{S_\beta}$ and $\alpha \leq \beta$, where $\overline{S_\beta}$ denotes the closure of $S_\beta$ in $\mathbb{R}^{n+1}$.*

**Definition 1.5.2.** *A closed set $M \subset \mathbb{R}^{n+1}$ is a* ***Whitney stratified set*** *with a Whitney stratification $\mathcal{S} = \{\mathcal{S}_\alpha\}_{\alpha \in I}$ if $M$ is a stratified set with all pairs of strata satisfying the* ***Whitney regularity conditions (a) and (b)****, defined as follows. Given a pair of strata $S_\alpha$ and $S_\beta$ with $S_\alpha \subset \overline{S_\beta}$, suppose $\{x_i\}$ is a sequence of points in $S_\beta$ converging to $y \in S_\alpha$, and $\{y_i\}$ a sequence of points in $S_\alpha$ also converging to $y$. Denote by $\tau$ the limit of the sequence of tangent spaces $T_{x_i}S_\beta$, and let $r$ denote the limiting secant line of the sequence $\{\overline{x_i y_i}\}$. Then:*

*(a) $T_y S_\alpha \subset \tau$, and*

*(b) $r \subset \tau$.*

Mather proved that Whitney's condition (b) implies condition (a) [**34**]. The sequences in Definition 1.5.2 may not have limits; however, by choosing subsequences, we may assume they converge.

In Chapter 5, we shall obtain *refinements* in the following sense of the stratifications of a collection of boundaries of regions and their medial axes.

**Definition 1.5.3.** *A* ***refinement*** $\mathcal{T} = \{\mathcal{T}_\gamma\}_{\gamma \in J}$ *of a stratification* $\mathcal{S} = \{\mathcal{S}_\alpha\}_{\alpha \in I}$ *is a stratification such that every stratum* $\mathcal{S}_\alpha$ *is a union of some collection of strata from* $\mathcal{T}$*.*

**1.5.2. Stratification of the critical set of family of distance squared functions.** In this section, under the hypotheses of Theorem 1.4.4, we describe the stratification $\mathscr{C}$ of the singular set of the family of distance squared functions $\sigma_\phi$.

For values of $n \leq 6$, there is a canonical ${}_r\mathcal{R}^+$-invariant stratification of the multijet space ${}_rJ^k(X, \mathbb{R})$ consisting of strata which are orbits under the action of ${}_r\mathcal{R}^+$ on $r$–multi $k$-jets of germs with simple singularities (see [**33**], [**37**]). Orbits under an algebraic group action form a Whitney stratification where they are locally finite, so the canonical stratification of jet space is Whitney. For a distance–generic embedding $\phi$, Theorem 1.4.1 implies that the multijet extension mapping

$$ {}_rj_1^k\sigma_\phi : X^{(r)} \times \mathbb{R}^{n+1} \to {}_rJ^k(X, \mathbb{R}) $$

is transverse to this stratification for sufficiently high $k$. This implies that the pull-back of the stratification to $X^{(r)} \times \mathbb{R}^{n+1}$ under ${}_rj_1^k\sigma_\phi$ is also Whitney stratified with strata of the form ${}_rj_1^k\sigma_\phi^{-1}(W_i)$, where $W_i$ is a stratum in ${}_rJ^k(X, \mathbb{R})$ [**34**]. Since the stratification of ${}_rJ^k(X, \mathbb{R})$ satisfies the boundary condition, so does the pull-back of the stratification; that is, if $W_j$ belongs to the closure of the stratum $W_i$ in ${}_rJ^k(X, \mathbb{R})$, then ${}_rj_1^k\sigma_\phi^{-1}(W_j)$ belongs to the closure of ${}_rj_1^k\sigma_\phi^{-1}(W_i)$.

By the uniqueness theorem for versal unfoldings of multigerms (Theorem 1.2.11), $\sigma_\phi$ is isomorphic to the unfolding $F \times \mathrm{id}_{n+1-p}$, where $F$ is a $p$–parameter miniversal unfolding. The following local model involving the singular set of the miniversal unfolding $F$ allows one to determine the explicit relationship between the stratification on the medial axis $M$ and the stratification on $\mathcal{B}$ to which it corresponds. Let $S_i = \{x_1, \ldots, x_r\}$ with each $x_i \in \mathbb{R}^n$, and suppose

$$F : (\mathbb{R}^n \times \mathbb{R}^p, (S, 0)) \to (\mathbb{R}, 0),\ F_w(x) = F(x, w) \tag{1.13}$$

is a versal unfolding of the multigerm $F_0 = f$ of singularity type $\mathcal{A}_\beta$. By the openness of versality (Theorem 1.2.12), since $F$ is versal in a neighborhood of $(S, 0)$, nearby germs of $f$ are also versally unfolded. Consider the real algebraic set

$$\{(x_1, \ldots, x_k, w) : F_w \text{ has a critical point at } x_j, F(x_j, w) = y\ \forall\ j\} \tag{1.14}$$

for different values of $k$. It is a real algebraic set since $F$ is versal and it is defined by the following algebraic conditions on polynomials:

$$\frac{\partial F_w}{\partial x}(x)\bigg|_{x=x_j} = 0,\ F(x_j, w) - y = 0 \text{ for } j = 1, \ldots, k.$$

Therefore, it is Whitney stratified with the stratification inherited from the pull-back of the canonical stratification of jet space.

For each set of the form in (1.14), its projections onto both $\mathbb{R}^n$ and $\mathbb{R}^p$ yield semi-algebraic sets by the Tarski-Seidenberg Theorem [**27**]; therefore, they are also Whitney stratified by singularity type of $F_w$. Moreover, the combined images of these projections will provide the local models for the stratifications of $\mathcal{B}$ and $M$, respectively, since the singular sets of $\sigma_\phi$ and $F' := F \times \mathrm{id}_{n+1-p}$ are locally diffeomorphic.

The versality theorem ensures that the projection of the stratum of the singular set of $\sigma_\phi$ corresponding to $\mathcal{A}_\beta$ singularity type to the parameter space $\mathbb{R}^{n+1}$ will be smooth. Let $\chi_{\mathcal{A}_\beta} \subset M$ denote this projection. In addition, we can project the stratum to $X$, which is diffeomorphic to $\mathcal{B}$ under the diffeomorphism $\phi$ provides. So, let $\Sigma_{\mathcal{A}_\beta} \subset \mathcal{B}$ denote the projection to $\mathcal{B}$ of the stratum for $\mathcal{A}_\beta$ singularity type. For an abstract versal unfolding, it need not be the case that the projection onto the manifold space is an embedding. One way to see that the $\Sigma_{\mathcal{A}_\beta}$ stratum on $\mathcal{B}$ is smooth is that it is the image of the $\chi_{\mathcal{A}_\beta}$

stratum on $M$ under a *radial flow* that is smooth on the strata of the medial axis; see Section 2.3 in Chapter 2.

**1.5.3. Explicit stratifications of $M$ and $\mathcal{B}$ in $\mathbb{R}^3$.** In this section, we determine the local structure of the stratifications of $M$ and $\mathcal{B}$ in $\mathbb{R}^3$ using the generic local normal forms of $\sigma_\phi$ given in Theorem 1.4.4.

First, we determine the local models of the $\chi_{A_1^k}$ strata on $M$ and the $\Sigma_{A_1^k}$ strata on $\mathcal{B}$ for $k = 2, 3, 4$. Let $F = (F_1, \ldots, F_k)$ be the standard $\mathcal{R}^+-$miniversal unfolding of the $A_1^k$ multigerm, i.e.,

$$
\begin{aligned}
F_1 &= (x_1^{(1)})^2 + (x_1^{(2)})^2 + u_1, \\
&\vdots \\
F_{k-1} &= (x_{k-1}^{(1)})^2 + (x_{k-1}^{(2)})^2 + u_{k-1}, \\
F_k &= (x_k^{(1)})^2 + (x_k^{(2)})^2.
\end{aligned}
$$

For $j = 1, \ldots, k$, $F_j$ has a critical point at $x_j^{(1)} = x_j^{(2)} = 0$ with critical value $u_j$ for $j < k$ and critical value 0 for $F_k$. The $k$ functions have equal minima when $u_1 = \ldots = u_{k-1} = 0$, and graphing this gives the $A_1^k$ stratum. To obtain the entire local model, we equate $2 \leq j < k$ of the critical values and set them less than or equal to the remaining critical values. By Theorem 1.2.11, $\sigma_\phi$ is isomorphic to $F' := F \times \mathrm{id}_{3-k}$. The codimension in $\mathbb{R}^3$ of the $\chi_{A_1^k}$ stratum is $k - 1$, the extended $\mathcal{R}^+-$codimension of the multigerm.

For $k = 2$, the local models of the $\chi_{A_1^2}$ and $\Sigma_{A_1^2}$ strata are smooth sheets. When $k = 3$, the local model of the $\chi_{A_1^3}$ stratum on $M$ consists of three half-planes meeting along a curve, called a *Y-branch curve*. See Figure 1.1. The corresponding $\Sigma_{A_1^3}$ stratum on $\mathcal{B}$ consists of three smooth curves associated to the same branch curve on $M$. Finally, the local model of the $\chi_{A_1^4}$ stratum on $M$ is a single point, called a *6-junction point*, occurring at the intersection of four branch curves and six half-planes. See Figure 1.1. Using the radial flow, we conclude that the $\Sigma_{A_1^4}$ stratum on $\mathcal{B}$ consists of 4 points, each of which has three mutually transverse curves passing through it that correspond to three distinct branch curves on $M$.

Next, we determine the local models of the $\chi_{A_3}$ stratum on $M$ and the $\Sigma_{A_3}$ stratum on $\mathcal{B}$. Let

$$F = x_1^4 + u_1 x_1^2 + u_2 x_1 + x_2^2$$

be the standard 2–parameter $\mathcal{R}^+-$miniversal unfolding of an $A_3$ singularity. This function has a critical point provided that $4x_1^3 + 2u_1x_1 + u_2 = 0$ and $x_2 = 0$. A direct computation shows that, in order to obtain minima, we must have $u_2 = 0$ and $u_1 \leq 0$ (specifically, it turns out that $u_1 = -2x_1^2$ to have a minimum). The versal unfolding $\sigma$ is isomorphic to $F' = F \times \mathrm{id}_1$. Since $u_3$ is a free parameter in $\mathbb{R}^3$, the local model is a half-plane that is bounded by a curve, known as an *edge curve* on the medial axis. See Figure 1.1. An edge curve on the medial axis corresponds to the $\Sigma_{A_3}$ stratum on the boundary, which is a *crest curve* consisting of points such that the larger principal curvature in absolute value at each point is a maximum along the associated principal direction (see, e.g., [**9**]).

The local models of the $\chi_{A_1A_3}$ stratum on $M$ and the corresponding $\Sigma_{A_1A_3}$ stratum on $\mathcal{B}$ are similarly established. See Figure 1.1 for the local model of the $\chi_{A_1A_3}$ stratum on $M$, called a *fin point*, which is a point at which a branch curve and an edge curve intersect and end.

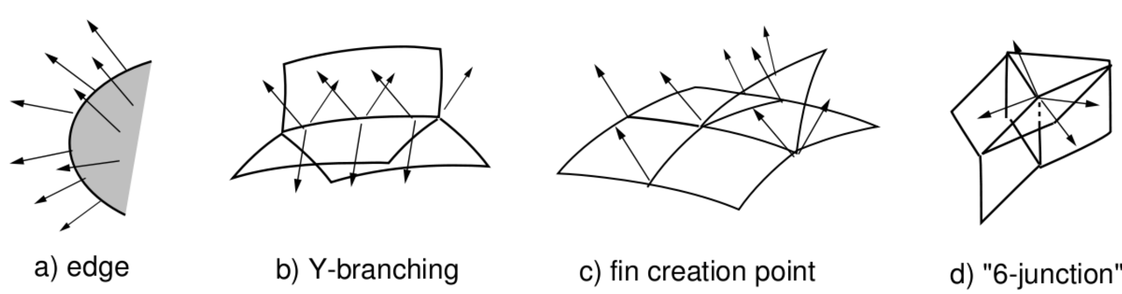


FIGURE 1.1. *Generic medial axis strata in* $\mathbb{R}^3$.

## 1.6. A note on the notions of codimension

In this dissertation, we shall refer to several types of codimension: the $\mathcal{R}_e^+$–codimension of a (germ or) multigerm of singularity type $\mathcal{A}_\beta$ as defined in Definition 1.2.6; the codimension as a submanifold of jet space of the orbit of the multigerm under the ${}_r\mathcal{R}^+$–action; the codimension in $\mathbb{R}^{n+1}$ of the medial axis stratum $\chi_{\mathcal{A}_\beta}$; and the codimension in $\mathcal{B}$ of the corresponding boundary stratum $\Sigma_{\mathcal{A}_\beta}$. In this section, we explain the relationships that exist among these various notions of codimension.

Suppose the family of distance squared functions $\sigma_\phi : (X \times \mathbb{R}^{n+1}, (S, w_0)) \to (\mathbb{R}, z)$ is a versal unfolding of the multigerm $\sigma_\phi(\cdot, w_0) : (X, S) \to (\mathbb{R}, z)$ of singularity type $\mathcal{A}_\beta$ with $|S| = r$. First, recall that the $\mathcal{R}_e^+$–codimension of a multigerm of singularity type $\mathcal{A}_\beta$, which we denote by $\mathcal{R}_e^+$–codim$(\mathcal{A}_\beta)$, equals the number of unfolding parameters in its miniversal unfolding. If $\text{codim}_{\mathbb{R}^{n+1}}(\chi_{\mathcal{A}_\beta})$ denotes the codimension in $\mathbb{R}^{n+1}$ of the corresponding medial axis stratum $\chi_{\mathcal{A}_\beta}$, we know from Section 1.5 that

$$\text{codim}_{\mathbb{R}^{n+1}}(\chi_{\mathcal{A}_\beta}) = \mathcal{R}_e^+\text{–codim}(\mathcal{A}_\beta). \tag{1.15}$$

Second, as mentioned in Section 1.5.2, there is a radial flow on the medial axis that provides a diffeomorphism between $\chi_{\mathcal{A}_\beta}$ and the corresponding stratum on the boundary $\Sigma_{\mathcal{A}_\beta}$. Therefore, since $\Sigma_{\mathcal{A}_\beta}$ and $\chi_{\mathcal{A}_\beta}$ have the same dimension, and since the codimension in $\mathcal{B}$ of the $\Sigma_{\mathcal{A}_\beta}$ stratum, denoted $\text{codim}_{\mathcal{B}}(\Sigma_{\mathcal{A}_\beta})$, equals $n - \dim(\Sigma_{\mathcal{A}_\beta})$, it follows from (1.15) that

$$\begin{aligned} \text{codim}_{\mathcal{B}}(\Sigma_{\mathcal{A}_\beta}) &= \text{codim}_{\mathbb{R}^{n+1}}(\chi_{\mathcal{A}_\beta}) - 1 \\ &= \mathcal{R}_e^+\text{–codim}(\mathcal{A}_\beta) - 1. \end{aligned} \tag{1.16}$$

Third, by the equivalence of versality and transversality, the mapping ${}_rj_1^k\sigma$ at $(S, w_0)$ is necessarily transverse to the orbit of ${}_rj_1^k\sigma(S, w_0)$ under the ${}_r\mathcal{R}^+$ group action on jet space. We let $W^\beta$ denote this orbit, and let $\text{codim}_{\text{JS}}(W^\beta)$ denote the codimension of $W^\beta$ as a submanifold of ${}_rJ^k(X, \mathbb{R})$. There is the following relation between the extended codimension and the jet space codimension (see, e.g., [**8**]):

$$\operatorname{codim}_{\mathrm{JS}}(W^{\beta}) = \mathcal{R}_e^+\text{–codim}(\mathcal{A}_{\beta}) + nr. \tag{1.17}$$

We first explain briefly why this relation holds in the $r = 1$ case. For $f \in \mathcal{E}_n$, we know that $\mathcal{R}_e^+$–codim$(f) = \dim_{\mathbb{R}}\mathcal{E}_n/J(f) - 1$, and

$$\mathfrak{m}_n J(f)/\mathfrak{m}_n^{k+1}$$

represents the orbit of the $k$–jet of $f$ in the jet space fiber $\mathfrak{m}_n/\mathfrak{m}_n^{k+1}$ [**10**]. Then the codimension of this orbit in $\mathfrak{m}_n/\mathfrak{m}_n^{k+1}$ is given by

$$\begin{aligned}
\dim_{\mathbb{R}} \left(\mathfrak{m}_n/\mathfrak{m}_n^{k+1}\right) \Big/ \left(\mathfrak{m}_n J(f)/\mathfrak{m}_n^{k+1}\right) &= \dim_{\mathbb{R}} \left(\mathfrak{m}_n/\mathfrak{m}_n J(f)\right) \\
&= \dim_{\mathbb{R}} \left(\mathfrak{m}_n/J(f)\right) + n \\
&= \dim_{\mathbb{R}} \left(\mathcal{E}_n/J(f)\right) - 1 + n \\
&= \mathcal{R}_e^+\text{–codim}(f) + n. \qquad (1.18)
\end{aligned}$$

The fact that $\dim_{\mathbb{R}} \left(\mathfrak{m}_n/\mathfrak{m}_n J(f)\right) = \dim_{\mathbb{R}} \left(\mathfrak{m}_n/J(f)\right)$ is due to the fact that $f$ has an isolated singularity and therefore the ideal $J(f)/\mathfrak{m}_n J(f)$ is of finite codimension, i.e., $\dim_{\mathbb{R}} \left(J(f)/\mathfrak{m}_n J(f)\right) = n$.

For a multigerm $f = (f_1, \ldots, f_r)$ with an $\mathcal{A}_{\beta}$ singularity, we consider the codimension of its orbit $W^{\beta}$ as a submanifold in $\left(\mathcal{E}_n/\mathfrak{m}_n^{k+1}\right)^r$, the multijet space fiber. Using (1.18) and the fact that the values of each of the $f_i$'s are necessarily equal, we have that the codimension of $W^{\beta}$ in $\left(\mathcal{E}_n/\mathfrak{m}_n^{k+1}\right)^r$ is

$$(r-1) + \sum_{i=1}^{r} (\mathcal{R}_e^+\text{–codim}(f_i) + n).$$

Then, since $\mathcal{R}_e^+$–codim$(f) = \displaystyle\sum_{i=1}^{r} \mathcal{R}_e^+$–codim$(f_i) + r - 1$ by (1.6), we see that the codimension of the orbit in multijet space is indeed given by (1.17).

CHAPTER 2

# Medial Geometry for a Single Region

## 2.1. Introduction

In this chapter, we present a survey of the results for the medial geometry of a single region as they apply to the Blum medial axis, with the goal in Chapter 6 of extending these results to configurations of multiple regions. The Blum medial axis is viewed as a Whitney stratified set $M$ on which is defined a (multivalued) radial vector field $U$, defined from points on $M$ to the points of tangency on the boundary. This is a special case of a skeletal structure. Earlier work focused on relating the differential geometry of the boundary of a region with the differential geometry of the medial axis involving derivatives of the radius function. Damon [**14**] [**15**] offered a more direct approach to studying the geometric properties of a region and its boundary via the medial geometry of the radial vector field on the medial axis. This involves two variations on the ordinary differential geometric shape operator and is related to the geometry of the region and its boundary via a radial flow, as we shall see in Sections 2.2 — 2.4.

Global geometric invariants of the regions and their boundaries are typically expressed by integrals of appropriate integrands. In Section 2.5, we explain how results from [**13**] allow one to compute integrals over the boundary of a region (Section 2.5.2) and over the region itself (Section 2.5.3) as integrals over the region's medial axis. Finally, in Section 2.6, we give an overview of the use of single region medial analysis in medical imaging. A number of medical and mathematical issues arise in extending medial analysis to multiple regions, and we will address these motivating questions through our development of the Blum medial linking structure.

## 2.2. The radial vector field

In this section, we recall the definition and properties of the radial vector field $U$ defined on the Blum medial axis $M$ of a compact, connected, orientable region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary $\mathcal{B}$.

Let $M_{\text{reg}}$ denote the set of regular points of $M$ that belong to the top–dimensional strata of $M$, and let $M_{\text{sing}}$ denote the union of all remaining strata. These singular points consist of (1) the set of non–edge points, (2) the set of edge points belonging to the boundary of $M$, denoted $\partial M$, and (3) the set of edge closure points belonging to the closure of the boundary of $M$, denoted $\overline{\partial M}$).

The *radial vector field $U$ on $M$* is a multivalued vector field with one value at each point $x_0 \in M$ for each of the associated tangency points on the boundary $\mathcal{B}$. Let $U = r \cdot \boldsymbol{u}$, where $\boldsymbol{u}$ is a multivalued unit vector field on $M$ and the *radial function* $r$ is a positive *multivalued function* on $M$, or a function which takes one value at each point for each of the values of $U$. A *smooth value* of the vector field $U$ at a smooth point $x_0 \in M$ is a neighborhood $V$ of $x_0$ together with a choice of values of the vector field on $V$ that constitute a smooth vector field on $V$.

The radial vector field satisfies the following properties (see [**40**] and [**14**]):

(1) (Behavior at smooth points) For any smooth point $x_0 \in M$, there are two values of $U$ that have the same length and make the same angle with the tangent space $T_{x_0}M$. The values of $U$ corresponding to one side of a neighborhood $V$ of $x_0$ form a smooth vector field.

(2) (Behavior at edge points) For any point $x_0 \in \partial M$, there is a single value of $U$ that points away from $M$ and is tangent to the smooth stratum containing $x_0$ in the closure.

(3) (Behavior at singular, non-edge points) Let $\overline{B}_\epsilon(x_0)$ denote a closed ball of radius $\epsilon$ centered at $x_0 \in M_{\text{sing}}$ with $x_0 \notin \partial M$, and let $M_\alpha$ be a *local component of* $x_0$, or a connected component of $\overline{B}_\epsilon(x_0) \cap M_{\text{reg}}$. Then both smooth values of $U$ on $M_\alpha$ smoothly extend to values on the stratum to which $x_0$ belongs. Moreover, there

corresponds to each value of $U$ at $x_0$ a connected component $C_i$ of $\overline{B}_\epsilon(x_0) \setminus M$, called a *local complementary component* of $M$ at $x_0$, into which $U(x_0)$ locally points in the sense defined in [**14**].

We end this section by recalling two shape operators on the medial axis that capture the medial geometry of the radial vector field.

**Definition 2.2.1.** *For a non-edge point $x_0 \in M$, $v \in T_{x_0}M$, and a choice of smooth value of the radial vector field $U = r\boldsymbol{u}$, define the **radial shape operator** $S_{rad}$ to be*

$$S_{rad}(v) = -proj_U \left( \frac{\partial \boldsymbol{u}}{\partial v} \right), \tag{2.1}$$

*where $proj_U$ denotes projection along $U$ onto the tangent space to the medial axis, $T_{x_0}M$. For an edge point $x_0 \in M$, $v \in T_{x_0}M$, and a smooth value of $U$ chosen on one side of $M$, define the **edge shape operator** $S_E$ to be*

$$S_E(v) = -proj' \left( \frac{\partial \boldsymbol{u}}{\partial v} \right), \tag{2.2}$$

*where $proj'$ denotes projection along $U$ onto $T_{x_0}\partial M \oplus \langle \boldsymbol{n} \rangle$, for $\boldsymbol{n}$ a unit normal vector field to $M$.*

Let $S_{\mathbf{v}}$ be the matrix representation of $S_{\text{rad}}$ with respect to the basis $\mathbf{v} = \{v_1, \ldots, v_n\}$ (resp., $S_{E\,\mathbf{v}}$ is the matrix representation of $S_E$ with respect to the basis $\{v_1, \ldots, v_n\}$ in the source, where $v_1, \ldots, v_{n-1}$ is a basis for $T_{x_0}\partial M$ and $v_n$ is a positive multiple of $\boldsymbol{u}$, and the basis $\{v_1, \ldots, v_{n-1}, \boldsymbol{n}\}$ in the target). Unlike the case of differential geometry, $S_{\text{rad}}$ is not self–adjoint; however, it can be diagonalized and the eigenvalues $\kappa_{r\,i}$ of $S_{\text{rad}}$ are the *principal radial curvatures*. Likewise, there are $n-1$ *principal edge curvatures*, which are the generalized eigenvalues $\kappa_{E\,i}$ of $(S_{E\,\mathbf{v}}, I_{n-1,1})$.

## 2.3. The local and global radial flows

Using the radial vector field $U$, in [**14**] Damon defines an outward flow, called the *radial flow*, from $M$ to the boundary $\mathcal{B}$ of the region $\Omega$. Since the radial vector field is multivalued and only defined on $M$, it is not possible to define a global radial flow from $M$ in the usual way. We first introduce the local version of the flow before explaining how to obtain a global version.

**Definition 2.3.1.** *The **local radial flow** $\psi$ in a neighborhood $V$ of a point $x_0 \in M$ for a smooth choice of the radial vector field $U$ is given by*

$$\psi : V \times [0,1] \to \mathbb{R}^{n+1}, \qquad (x,t) \mapsto \psi_t(x) = \psi(x,t) = x + tU(x). \tag{2.3}$$

We refer to $\psi_1$, which maps from a region on $M$ to the corresponding region on the boundary $B$, as the *radial map*.

In order to consider both sides of the medial axis simultaneously, we introduce the notion of the *double* of the medial axis $M$, on which a global version of the radial flow is defined.

**Definition 2.3.2.** *The **double** of the Blum medial axis $M$, denoted $\widetilde{M}$, is the set*

$$\widetilde{M} = \{(x, U') \in M \times \mathbb{R}^{n+1} \mid U' \textit{ is a value of } U \textit{ at } x\}.$$

As explained in [**14**], $\widetilde{M}$ may be given a topology in the following way. First, for $x_0 \in M_{\text{reg}}$ with a value $U(x_0)$ of the radial vector field and a neighborhood $V$ of $x_0$, a neighborhood of $(x_0, U(x_0)) \in \widetilde{M}$ is given by $(V \times \{U_0\}) \cap \widetilde{M}$, where $\{U_0\}$ denotes the values of a continuous extension $U_0$ of $U(x_0)$ to $V$. Next, for a neighborhood $V$ of a point $x_0 \in M_{\text{sing}}$ and a choice of radial vector $U(x_0)$ that points into some complementary component $C_i$, a neighborhood of $(x_0, U(x_0)) \in \widetilde{M}$ is the intersection of a set $(V' \cap \partial C_i) \times \{U_0\}$ with $\widetilde{M}$. Here, $V' \subset V$ is a neighborhood of $x_0$ in $\mathbb{R}^{n+1}$ and $\{U_0\}$ consists of values of a continuous extension of $U(x_0)$ to $V' \cap \partial C_i$. Damon referred to the neighborhoods in $\widetilde{M}$ as *abstract neighborhoods*; see Figure 2.1 for an example.

There is a canonical line bundle $N$ on $\widetilde{M}$ which is spanned at a point $(x_0, U_0) \in \widetilde{M}$ by $U_0$. This is a trivial bundle and there are half–neighborhoods of the 0–section $N_\epsilon =$

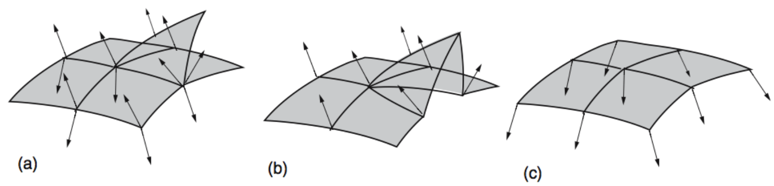


FIGURE 2.1. *Example of an abstract neighborhood. For (a), the local structure of a 3D medial axis near a fin point. For (b) and (c), the abstract neighborhoods of the two points in $\widetilde{M}$ corresponding to the fin point.*

$\{tU_0 \in N : 0 \leq t \leq \epsilon\}$. Then the *global radial flow* is the map

$$\widetilde{\psi} : N \to \mathbb{R}^{n+1}, \tag{2.4}$$
$$(x_0, tU_0) \mapsto x_0 + tU_0.$$

First, it is proven in [**14**] that there is an $\epsilon > 0$ so that the radial flow $N_\epsilon \to \mathbb{R}^{n+1}$ is a diffeomorphism onto a "tubular neighborhood" of $M$. In order to establish the global nonsingularity of the radial flow, we recall three conditions from [**14**] that the Blum medial axis satisfies. First, if $dr$ denotes the gradient of the radius function, the *compatibility condition* at a point $x_0 \in M$ states that the *compatibility 1-form*

$$\eta_U(v) = v \cdot U + dr(v) \tag{2.5}$$

vanishes at $x_0$ for any $v \in T_{x_0}M$. The compatibility condition ensures that the radial vector field is orthogonal to the boundary of the region [**14**]. Second, at any point $x_0 \in M$ with $x_0 \notin \partial M$, the radial shape operator $S_{\text{rad}}$ satisfies the following *radial curvature condition*:

$$r < \min\left\{\frac{1}{\kappa_{r_i}}\right\} \text{ for all positive principal radial curvatures } \kappa_{r_i} \text{ of } S_{\text{rad}}. \tag{2.6}$$

Third, at any $x_0 \in \overline{\partial M}$, the edge shape operator $S_E$ satisfies the following *edge condition*:

$$r < \min\left\{\frac{1}{\kappa_{E\,i}}\right\} \text{ for all positive principal edge curvatures } \kappa_{E\,i} \text{ of } S_E. \tag{2.7}$$

In [**14**], Damon showed that, given the three conditions listed above, the global radial flow is a homeomorphism $N_1 \to \Omega$ (fibering $\Omega \setminus M$ with the level sets of the flow) which is a local diffeomorphism from points $(x_0, U)$ with $x_0 \in M_{\text{reg}}$ and a local piecewise smooth homeomorphism on a neighborhood of a point $(x_0, U)$ with $x_0 \in M_{\text{sing}}$ [**14**].

In Chapter 6, we will introduce an extension of the radial flow called the *linking flow*. The integrability of the radial flow will apply to the integrability of the linking flow.

### 2.4. Medial geometry via the radial and edge shape operators

In this section, we recall how the radial and edge shape operators evolve under the radial flow. This enables one to find a matrix representation for the radial shape operator for $\mathcal{B}_t$, the level hypersurface of the radial flow at time $t$, in terms of the shape operator on $M$. In Chapter 6, we extend this notion to determine the behavior of the radial shape operator under the linking flow.

Damon proved the following two propositions in [**15**] (Propositions 2.1 and 2.3, respectively).

**Proposition 2.4.1.** *Let $x_0 \in M_{reg}$ with a smooth value of $U$ and a basis $\boldsymbol{v} = \{v_1, \ldots, v_n\}$ of $T_{x_0}M$. Suppose $\dfrac{1}{tr}$ is not an eigenvalue of the radial shape operator $S_{\boldsymbol{v}}$ at $x_0$. Then, if $\psi_t(x_0) = x_0'$ and $\boldsymbol{v}'$ denotes the image of $\boldsymbol{v}$ under $d\psi_t(x_0)$, the radial shape operator $S_{\boldsymbol{v}'t}$ for $\mathcal{B}_t$ at $x_0'$ is given by*

$$S_{\boldsymbol{v}'t} = (I - tr \cdot S_{\boldsymbol{v}})^{-1} S_{\boldsymbol{v}}. \tag{2.8}$$

**Proposition 2.4.2.** *Let $x_0 \in \partial M$ with a smooth value of $U$ (corresponding to one side of $M$), and let $\boldsymbol{v} = \{v_1, \ldots, v_{n-1}, v_n\}$, where $v_n = \boldsymbol{u}$, be a basis of $T_{x_0}M$. Suppose $\dfrac{1}{tr}$ is not a generalized eigenvalue of $(S_{E\boldsymbol{v}}, I_{n-1,1})$. Then, if $\psi_t(x_0) = x_0'$ and $\boldsymbol{v}' = \{v_1, \ldots, v_{n-1}, \boldsymbol{n}\}$ denotes the image of $\boldsymbol{v}$ under $d\psi_t(x_0)$, the radial shape operator $S_{\boldsymbol{v}'t}$ for $\mathcal{B}_t$ at $x_0'$ is given by*

$$S_{\boldsymbol{v}'t} = (I_{n-1,1} - tr \cdot S_{E\boldsymbol{v}})^{-1} S_{E\boldsymbol{v}}. \tag{2.9}$$

**Remark 2.4.3.** When $t = 1$ in the above propositions, one obtains a matrix representation for the regular differential geometric shape operator on the boundary $\mathcal{B}$ in terms of the radial and edge shape operators on $M$. This makes it possible to explicitly relate the principal radial and edge curvatures with the ordinary principal curvatures on the boundary (see Theorem 3.2 and Corollary 3.9 in [**15**]).

### 2.5. Blum medial axis as a measure space

Global invariants of a region $\Omega$ or its boundary $\mathcal{B}$ are expressed by integrals over these spaces. We explain how such invariants can be computed from the medial geometry as integrals over $\widetilde{M}$ as defined in Definition 2.3.2. In Chapter 6, we shall extend these results to integrals over the complements of multiple regions.

**2.5.1. Introduction to integration over the medial axis.** We first explain how to integrate a multivalued function $g$ over the medial axis $M$, as introduced by Damon in [**13**]. Such a multivalued function $g$ lifts to a well-defined function $\widetilde{g} = g \circ \pi$ on $\widetilde{M}$, where $\pi$ denotes the natural projection $\pi : \widetilde{M} \to M$. A *multivalued measurable* (resp., *integrable* or *continuous*) *function* $g$ on $M$ is a multivalued function such that $\widetilde{g}$ is measurable (resp., integrable or continuous) on $\widetilde{M}$.

Damon used the Riesz Representation Theorem to prove the existence of a unique regular positive Borel measure $dM$ on $\widetilde{M}$ (Proposition 2.2 in [**13**]). Then, for a multivalued continuous function $g$ on $M$, the medial integral of $\widetilde{g}$ on $\widetilde{M}$ is given by integration with respect to $dM$. In the Blum case, this measure

$$dM = \rho\, dV = \boldsymbol{u} \cdot \boldsymbol{n}\, dV \tag{2.10}$$

is defined on the medial axis itself and is referred to as the *medial measure*. As above, $\boldsymbol{n}$ denotes the unit normal vector field on $M$, $\boldsymbol{u}$ the unit normal radial vector field, and $dV$ the $n$-dimensional Riemannian volume measure. (The measure $dM$ has the effect of correcting for the non-orthogonality of the radial vector field $U$ to $M$.) Consequently, regions where the radial vector field is nearly orthogonal to the medial axis contribute more to the value of the integral than places where the radial vectors are nearly tangent to the medial axis (e.g., near the edge or regions with small protrusions).

**2.5.2. Computing boundary integrals as medial integrals.** The next three results demonstrate how to compute integrals of functions over a boundary $\mathcal{B}$ or over a region $\Gamma \subset \Omega$ as integrals over the medial axis (see [**13**], Theorem 1, Corollary 2, and Theorem 3, respectively).

**Theorem 2.5.1.** *Suppose $(M, U)$ is the Blum medial axis and radial vector field $U$ for a region $\Omega$ with smooth boundary $\mathcal{B}$. Let $f : \mathcal{B} \to \mathbb{R}$ be a Borel measurable function that is integrable with respect to $dV$, the Riemannian volume measure. Then*

$$\int_{\mathcal{B}} f\, dV = \int_{\widetilde{M}} f(x + U(x)) \cdot det(I - rS_{rad})\, dM. \tag{2.11}$$

Note that $f(x + U(x)) = f(\widetilde{\psi}_1(x))$, which is a function on $\widetilde{M}$, descends to a multivalued function on $M$.

**Corollary 2.5.2.** *If $R$ denotes a Borel measurable subset of $\mathcal{B}$ and $f : R \to \mathbb{R}$ is as in Theorem 2.5.1, then letting $\widetilde{R} = \psi_1^{-1}(R)$, we have*

$$\int_{R} f\, dV = \int_{\widetilde{R}} f(x + U(x)) \cdot det(I - rS_{rad})\, dM. \tag{2.12}$$

Replacing the function $f$ in Theorem 2.5.1 with the function that is identically equal to 1, we obtain the formula for the volume of $\mathcal{B}$ written as a medial integral.

**Corollary 2.5.3.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a compact region with smooth boundary $\mathcal{B}$ and Blum medial axis $M$. The $n$-dimensional volume of $\mathcal{B}$ is given by*

$$vol(\mathcal{B}) = \int_{\widetilde{M}} det(I - rS_{rad})\, dM. \tag{2.13}$$

**2.5.3. Computing integrals on regions as medial integrals.** Let $g : \Omega \to \mathbb{R}$ be a Borel measurable and Lebesgue integrable function. Using the notation of [**13**], let

$$g_1(x, t) = (g \circ \widetilde{\psi})(x, t) = g(x + tU(x)),$$

and $\widetilde{g}$ is the multivalued function on $M$ given by

$$\widetilde{g}(x) = \int_0^1 g_1(x,t) \cdot \det(I - trS_{\text{rad}})\, dt.$$

The next result (Theorem 6 in [**13**]) establishes how to integrate $g$ over the region $\Omega$ as a medial integral.

**Theorem 2.5.4.** *Suppose $(M, U)$ is the Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$. Let $g : \Omega \to \mathbb{R}$ be Borel measurable and integrable with respect to Lebesgue measure. Then, $\widetilde{g}$ is defined for almost every $x \in \widetilde{M}$, integrable on $\widetilde{M}$, and*

$$\int_\Omega g\, dV = \int_{\widetilde{M}} \widetilde{g} \cdot r\, dM. \tag{2.14}$$

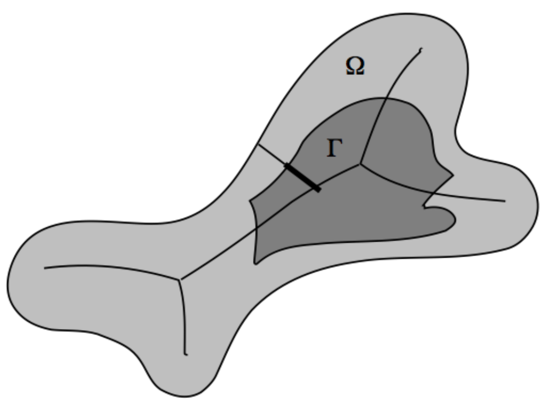


FIGURE 2.2. *Computing integrals over regions $\Gamma \subset \Omega$ as medial integrals.*

Given a Borel measurable function $g : \Omega \to \mathbb{R}$, it is possible to integrate $g$ over a smaller region $\Gamma \subset \Omega$, as in Figure 2.2. One must first compose $g$ with the radial flow, compute a line integral over the portion of the radial line within $\Gamma$, then integrate the resulting function over the medial axis, as $M$ parametrizes such lines [**13**]. This Crofton-type formula (Corollary 7 in [**13**]) is given in the next theorem.

**Theorem 2.5.5.** *Suppose $(M, U)$ is the Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$. Let $\Gamma \subset \Omega$ be a Borel measurable region, and let $g : \Gamma \to \mathbb{R}$ be a Borel* *measurable and Lebesgue integrable function. If*

$$\widetilde{g}_\Gamma(x) = \int_0^1 \chi_\Gamma \cdot g(x + tU(x)) \cdot det(I - trS_{rad})\, dt,$$

*then* $\widetilde{g}_\Gamma$ *is defined for almost all* $x \in \widetilde{M}$*, integrable on* $\widetilde{M}$*, and*

$$\int_\Gamma g\, dV = \int_{\widetilde{M}} \widetilde{g}_\Gamma \cdot r\, dM. \tag{2.15}$$

**2.5.4. Examples of integration.** In this section, we explicitly compute the formulas for area and volume of a region $\Omega$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ as integrals over $M$. These formulas generalize the classical formulas of Weyl for volumes of tubes and Steiner's formula.

**Example 2.5.6** ($n = 2$)**.** Suppose $\Omega \subset \mathbb{R}^2$ is a smooth compact region with smooth boundary $\mathcal{B}$ and Blum medial axis $M$. In $\mathbb{R}^2$, the radial shape operator is simply multiplication by $\kappa_r$, the radial curvature of $M$. Then the following integral is defined*:*

$$\alpha = \int_0^1 det(I - trS_{rad})\, dt = \int_0^1 (1 - tr\kappa_r)\ dt = 1 - \frac{1}{2} r\kappa_r. \tag{2.16}$$

Using the above formula, we determine the area of $\Omega$ to be

$$\begin{aligned} area(\Omega) &= \int_{\widetilde{M}} \alpha \cdot r\, dM \\ &= \int_{\widetilde{M}} r\, dM - \frac{1}{2} \int_{\widetilde{M}} r^2 \cdot \kappa_r\, dM. \end{aligned} \tag{2.17}$$

**Example 2.5.7** ($n = 3$)**.** As above, suppose $\Omega \subset \mathbb{R}^3$ is a smooth compact region with smooth boundary $\mathcal{B}$ and Blum medial axis $M$. Since

$$det(I - trS_{rad}) = 1 - tr \cdot trace(S_{rad}) + t^2 r^2 \cdot det(S_{rad}),$$

we have

$$\alpha = \int_0^1 det(I - trS_{rad})\, dt = 1 - r\, H_{rad} + \frac{1}{3} r^2\, K_{rad},$$

where $H_{\text{rad}} = \dfrac{1}{2}\text{trace}(S_{\text{rad}})$ and $K_{\text{rad}} = \det(S_{\text{rad}})$. Hence, we determine that the volume of $\Omega$ is given by the following formula:

$$
\begin{aligned}
\mathit{volume}(\Omega) &= \int_{\widetilde{M}} \alpha \cdot r \, dM \\
&= \int_{\widetilde{M}} r \, dM - \int_{\widetilde{M}} r^2 \cdot H_{rad} \, dM + \frac{1}{3} \int_{\widetilde{M}} r^3 \cdot K_{rad} \, dM. \qquad (2.18)
\end{aligned}
$$

**Remark 2.5.8.** Although the medial axis is stable under sufficiently small $C^\infty$ perturbations of the boundary, it fails to be stable under small $C^1$ perturbations — namely, a small change on the boundary may produce a large set-theoretic change on the medial axis. Nevertheless, area/volume undergoes only small changes under sufficiently small perturbations. In Chapter 6, we shall introduce several measures of comparison for a collection of regions which involve computing the area and volume of regions extending into the complement as medial integrals.

## 2.6. Medial representations in medical imaging

In this section, we describe how medial representations of a single object or region are utilized by computer scientists in the field of medical imaging. We also describe work involving multiple regions and identify a number of issues relating to multi–region analysis, both medical and mathematical in nature, that motivated the work in this dissertation.

A discretized version of the Blum medial axis was introduced in [**42**] by Pizer, et al., who referred to it as a discrete "m-rep" or medial representation of an object. It is composed of a finite collection of medial atoms consisting of a center point and two "spokes" or radial vectors. Examples of discrete m–reps in 2 and 3 dimensions appear in Figures 2.3 and 2.4, respectively.

Pizer, et al. introduced a *deformable model approach* to shape analysis in medical images. This approach requires the initial selection of a medial model or template to fit to an object or organ in an image, with every other instance of the image then defined as a diffeomorphism applied to the initial model. Obtaining a medial model from a grayscale

FIGURE 2.3. *2D discrete m–rep used for segmentation of the brainstem* [**42**].

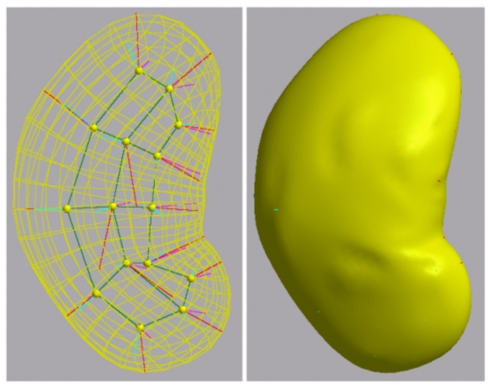

FIGURE 2.4. *Discrete m–rep of the kidney and 3D rendering (courtesy of Medical Image Display and Analysis Group at UNC).*

medical image provides a means of *segmenting* or locating the organ in the image; for an example, see Figure 2.3 which depicts a segmentation of the brainstem. Moreover, the deformable model approach allows for comparison of the anatomical structure of organs based on their medial representations using statistical analysis of populations of regions performed directly on the discrete m–reps. Such analysis requires a correspondence across images of the same patient on different days or times, or across images of the same organs in different patients, and is used for such purposes as discriminating between diseased and healthy patients. From the medical perspective, another primary objective of such analysis is in treatment planning to ensure that the accurate amount of treatment (e.g., radiation) is delivered to the precise destination.

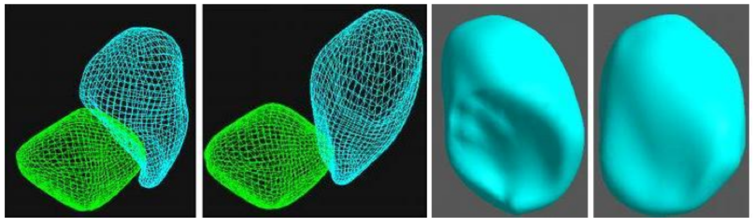

FIGURE 2.5. *Example of discrete m–reps of a complex of organs. On the left, discrete m–reps of the prostate and bladder. On the right, bladder images produced from the m–reps demonstrating the influence of the prostate on the bladder* [**26**].

Pizer and other members of the Medical Image Display and Analysis Group at the University of North Carolina have begun to apply the techniques of single region medial analysis to multiple objects or regions, such as complexes of organs in the body. We now describe a number of medical and mathematical issues that arise in the context of multiple region shape analysis.

First, fundamental questions from the single region setting carry over to multi–region shape analysis, such as:

(1) How may one perform statistical analysis on images of organ complexes in order to effectively analyze the images and aid in treatment planning?

Moreover, moving from shape analysis of one object to a collection of objects involves the examination of the following:

(2) What relationships exist within a collection of objects, including any influences that objects may exert over other objects?

For example, when the bladder fills, the nearby prostate presses on it and changes its shape as illustrated in Figure 2.5.

Another issue that arises in the multi–object context is that some organs, such as the prostate, have a very low degree of across–boundary intensity contrast and are therefore difficult to accurately segment in an image. It is possible to use nearby objects in the body such as the pubic bones, which have a higher across–boundary contrast, to statistically

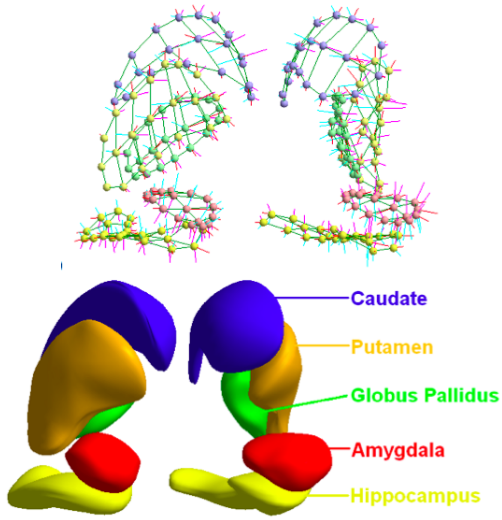


Figure 2.6. *A second example of a multi–object complex. Image of subcortical brain structures constructed from multiple m-reps used in a study of differences in brain structures between autistic and typically developing children* [**24**].

predict the location of the prostate. This issue brings up the following mathematical question:

(3) How may one obtain a correspondence between regions on nearby objects that should to some degree be statistically correlated based on their proximity to one another, as well as examine variations in this correspondence across populations?

Presently, Pizer, et al. employ user-based identification of object or region closeness (see, e.g., [**41**], [**30**], [**26**]), so one objective in extending medial analysis to multiple regions is to aid in rigorizing the choice of correspondence between neighboring regions.

Another issue relates to the fact mentioned earlier that organs or portions of objects in the body may undergo shape or position changes based on the influences of other nearby organs:

(4) Which objects or regions on objects are most significant within a given collection?

Furthermore, the current deformable model approach used by Pizer, et al. to deform a complex of objects in an image is based on different orderings that the user places on the objects and sections of objects, involving properties such as image intensity and geometric stability [**24**]. The deformations are comprised of various degrees of locality or scale levels, including global deformations applied to every discrete m–rep in an image to deformations within a portion of a single m–rep. This raises the question of how can one make the following notion mathematically precise:

(5) How may one study relations among objects or sections of objects at the same scale level?

In Chapter 6, we shall turn to the topic of how the medial linking structure may be used to address such questions from both mathematics and medical image analysis.

CHAPTER 3

# Blum Medial Linking Structure as Extension of Medial Analysis to Multiple Regions

## 3.1. Introduction

In this chapter, we extend the analysis of the Blum medial axis of a single region to multiple disjoint regions by introducing the *Blum medial linking structure*. This structure is designed to enable us to study the geometry of a collection of regions relative to one another by capturing the relationships between their medial axes and their interactions with the exterior medial axis of the complementary region.

We begin Section 3.2 by stating our genericity assumptions and their consequences, then introduce the notion of medial linking in Section 3.3. In Section 3.4, we expand upon the classification of Mather and Yomdin described in Chapter 1 by classifying the generic forms of medial linking in 2 and 3 dimensions, deferring the proofs of the transversality theorem and transversality conditions that yield these results to Chapters 4 and 5, respectively. As a consequence of the classification theorems, we develop in Section 3.5 a fundamental component of the linking structure: labeled refinements of the Whitney stratifications of the boundaries and the medial axes to reflect interactions between regions. In Section 3.6, we define the final major piece of the linking structure, a collection of multivalued vector fields defined on the individual regions' medial axes that satisfy certain properties in relation to other regions' medial axes and the medial axis of the complementary region.

Finally, in Section 3.7, we combine the aforementioned ingredients to define the Blum medial linking structure associated to a collection of regions. Later, in Chapter 6, we shall apply the linking structure to address questions of interest from the perspectives of both mathematics and medical imaging.

### 3.2. Genericity assumptions

We begin with some initial genericity assumptions from the single region case which we shall further supplement for the multi-region setting. Let $X = \coprod_{i=1}^{q} X_i$, where each $X_i$ is a smooth, $n$-dimensional, compact, connected, orientable manifold. Let $\phi : X \hookrightarrow \mathbb{R}^{n+1}$ be a smooth embedding, and let $\phi|X_i = \phi_i : X_i \hookrightarrow \mathbb{R}^{n+1}$ for $i = 1, \ldots, q$. For each $i$, let $\mathcal{B}_i = \phi_i(X_i)$; by the Jordan-Brouwer Separation Theorem, $\mathcal{B}_i$ bounds a compact connected region in $\mathbb{R}^{n+1}$, which we denote by $\Omega_i$. We shall restrict our attention to the subset $\mathrm{DEmb}(X, \mathbb{R}^{n+1}) \subset \mathrm{Emb}(X, \mathbb{R}^{n+1})$ of smooth embeddings satisfying the condition that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. We begin with a simple lemma.

**Lemma 3.2.1.** *The set* $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ *is an open subset of* $\mathrm{Emb}(X, \mathbb{R}^{n+1})$ *in the* $\mathrm{C}^\infty$ *topology.*

Proof. Let $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$, where $\phi(X_i) = \mathcal{B}_i$ bounds a region $\Omega_i$ for every $i = 1, \ldots, q$, and let $\delta = \min_{i \neq j} d(\Omega_i, \Omega_j)$, where $d$ denotes the minimum distance from $\Omega_i$ to $\Omega_j$. Now, $\delta > 0$ since the $\Omega_i$ are compact, pairwise disjoint, and finite in number. For any $\epsilon > 0$ and for any $i = 1, \ldots, q$, let

$$\Omega_i^\epsilon = \{x \in \mathbb{R}^{n+1} : d(x, \Omega_i) < \epsilon\}.$$

Assume $\epsilon < \delta/4$. Observe that, by the triangle inequality and the definition of $\delta$, $\Omega_i^\epsilon \cap \Omega_j^\epsilon = \emptyset$ for $i \neq j$. If necessary, shrink $\epsilon$ to ensure that $\partial\Omega_i^\epsilon$ is smooth for every $i$. That this is possible follows from the fact that

$$\partial\Omega_i^\epsilon = \{x \in \mathbb{R}^{n+1} : d(x, \Omega_i) = \epsilon\}$$

is the image of $\mathcal{B}_i$ at time $\epsilon$ under the eikonal or grassfire flow in the complement, which is a diffeomorphism for $\epsilon$ sufficiently small.

Next, let

$$U = \{\psi \in \mathrm{Emb}(X, \mathbb{R}^{n+1}) : ||\phi - \psi||_{\mathrm{C}^0} < \epsilon\}.$$

We will show that $U \subset \mathrm{DEmb}(X, \mathbb{R}^{n+1})$, establishing that $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ is open in the $\mathrm{C}^0$ topology.

Let $\psi \in U$. By the Jordan-Brouwer Separation Theorem, $\psi(X_i) = \mathcal{B}'_i$ bounds a region $\Omega'_i$ for all $i = 1, \ldots, q$. To show that $\psi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$, it suffices to show that $\Omega'_i \subset \Omega^\epsilon_i$, since the collection of $\Omega^\epsilon_i$ is pairwise disjoint. For each $i$, $\mathcal{B}'_i \subset \Omega^\epsilon_i$ since $||\phi - \psi|| < \epsilon$; it remains to show that $\mathring{\Omega}'_i$, the interior of $\Omega'_i$, is also contained in $\Omega^\epsilon_i$. To establish this, we first prove the following claim.

**Claim 3.2.2.** *$\mathring{\Omega}'_i$ does not intersect $\partial\Omega^\epsilon_i$.*

PROOF. Suppose $\mathring{\Omega}'_i \cap \partial\Omega^\epsilon_i \neq \emptyset$. Consider the set $W := \mathring{\Omega}'_i \cup (\mathbb{R}^{n+1} \setminus \Omega^\epsilon_i)$, which is connected since both $\mathring{\Omega}'_i$ and $\mathbb{R}^{n+1} \setminus \Omega^\epsilon_i$ are connected and there exists a point in their intersection (since $\partial\Omega^\epsilon_i \subset \mathbb{R}^{n+1} \setminus \Omega^\epsilon_i$). $W$ does not intersect $\mathcal{B}'_i$ since $\mathring{\Omega}'_i \cap \mathcal{B}'_i = \emptyset$ and $\mathcal{B}'_i \subset \Omega^\epsilon_i$. By the Jordan-Brouwer Separation Theorem applied to $\mathcal{B}'_i$, the unbounded set $W$ is contained in the unbounded component of $\mathbb{R}^{n+1} \setminus \mathcal{B}'_i$. But this contradicts the fact that $\mathring{\Omega}'_i$ is, by definition, in the bounded component. $\square$

Applying the Jordan-Brouwer Separation Theorem to $\partial\Omega^\epsilon_i$, $\mathbb{R}^{n+1} \setminus \partial\Omega^\epsilon_i$ consists of exactly two open connected components: $\Omega^\epsilon_i$ and $\mathbb{R}^{n+1} \setminus \overline{\Omega}^\epsilon_i$, where $\overline{\Omega}^\epsilon_i$ denotes the closure of $\Omega^\epsilon_i$. Since $\mathring{\Omega}'_i$ is connected and does not intersect $\partial\Omega^\epsilon_i$ by the claim, it is a subset of one of the two components of $\mathbb{R}^{n+1} \setminus \partial\Omega^\epsilon_i$. If $\mathring{\Omega}'_i \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}^\epsilon_i$, then $\mathcal{B}'_i \subset \Omega'_i \subset \mathbb{R}^{n+1} \setminus \Omega^\epsilon_i$, a contradiction to the fact that $\mathcal{B}'_i \subset \Omega^\epsilon_i$.

Therefore, $\Omega'_i \subset \Omega^\epsilon_i$ for every $i$. Since $\Omega^\epsilon_i \cap \Omega^\epsilon_j = \emptyset$ for $i \neq j$, it follows that $\Omega'_i \cap \Omega'_j = \emptyset$ for $i \neq j$, as well. Thus, $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ is open in the $\mathrm{C}^0$ topology, ensuring that it is also open in the finer $\mathrm{C}^\infty$ topology.

$\square$

By the result of Looijenga given in Theorem 1.4.1 in Chapter 1, there is a residual set of embeddings $\mathscr{R} \subset \mathrm{Emb}(X, \mathbb{R}^{n+1})$ that satisfy the notion of distance-genericity. A second lemma establishes that the subset of these embeddings in which we are interested is itself a residual set.

**Lemma 3.2.3.** *The set* $\mathrm{DEmb}(X, \mathbb{R}^{n+1}) \cap \mathscr{R}$ *is a residual subset of* $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$*, and is dense in* $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$*.*

Proof. Since $\mathscr{R}$ is residual, let $\mathscr{R} = \bigcap_j \mathcal{O}_j$ be a countable intersection of open dense subsets in $\mathrm{Emb}(X, \mathbb{R}^{n+1})$. Since $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ is open, for any $j$, the set $\mathrm{DEmb}(X, \mathbb{R}^{n+1}) \cap \mathcal{O}_j$ is also open in $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$. Furthermore, for any $j$, the set $\mathrm{DEmb}(X, \mathbb{R}^{n+1}) \cap \mathcal{O}_j$ is dense in $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$. This is because, given $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$, any neighborhood of $\phi$ that is open in $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ is open in $\mathrm{Emb}(X, \mathbb{R}^{n+1})$, and therefore must intersect $\mathcal{O}_j$ since $\mathcal{O}_j$ is dense in $\mathrm{Emb}(X, \mathbb{R}^{n+1})$. Hence, $\bigcap_j \mathrm{DEmb}(X, \mathbb{R}^{n+1}) \cap \mathcal{O}_j$ is residual in $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ since it is a countable intersection of open dense subsets of $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$. By Lemma 3.2.1, $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ is an open subset of the Baire space $\mathrm{Emb}(X, \mathbb{R}^{n+1})$, and therefore is itself a Baire space [**39**]. Since a residual subset of a Baire space is dense, it follows that $\mathrm{DEmb}(X, \mathbb{R}^{n+1}) \cap \mathscr{R}$ is dense in $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$. □

From now on, when we refer to a *generic* embedding, we shall mean an element of a residual subset of $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ satisfying a precise condition involving transversality to certain submanifolds of jet space. This transversality condition will be made explicit in Section 5.3 in Chapter 5. For $n \leq 6$, the genericity conditions imply that each disjoint region $\Omega_i$ will have a Blum medial axis, $M_i$, with the generic local normal forms and properties given by Mather and Yomdin and described in detail in Chapters 1 and 2. The genericity assumptions further guarantee that the unbounded complementary region will also have a generic Blum medial axis, denoted $M_0$.

## 3.3. Definition of medial linking

In this section, we shall introduce the concept of *medial linking* for a collection of disjoint regions $\{\Omega_i\}_{i=1}^q$ with smooth boundaries $\partial\Omega_i = \mathcal{B}_i$ defined by a generic embedding

$\phi : X \to \mathbb{R}^{n+1}$ for $\phi \in \text{DEmb}(X, \mathbb{R}^{n+1})$. Let $\mathcal{B} = \coprod_{i=1}^{q} \mathcal{B}_i$ and $\Omega = \coprod_{i=1}^{q} \Omega_i$; both $\mathcal{B}$ and $\Omega$ are closed because the boundary hypersurfaces and the regions they bound are compact. Also, let $\Omega_0 = \overline{\Omega^c}$, the closure of the region complementary to $\Omega$. By the General Separation Theorem [**25**], since $\mathcal{B}$ has $q$ components, the complement $\mathbb{R}^{n+1} \setminus \mathcal{B}$ has $q+1$ components; it follows that there is only one unbounded connected component of the complement. We shall let $\mathring{\Omega}_0$ denote the interior of this unbounded component of $\Omega_0$.

To each hypersurface $\mathcal{B}_i$, $i = 1, \ldots, q$, we assign a parametrized family of distance squared functions

$$\sigma_i : \mathcal{B}_i \times \mathring{\Omega}_i \to \mathbb{R}, (x, w) \mapsto ||x - w||^2.$$

This defines, for each point $w \in \mathring{\Omega}_i$, a smooth map

$$\sigma_i(\cdot, w) : \mathcal{B}_i \to \mathbb{R},\ x \mapsto \sigma_i(x, w) = ||x - w||^2.$$

As explained in Section 1.4.2 in Chapter 1, using distance squared rather than distance ensures smoothness and preserves the singularity theoretic information since both the distance function and the distance squared function defined on $\mathcal{B}_i \times \mathring{\Omega}_i$ are nonzero with the same types of critical points.

Additionally, we define a family of functions that measures the squared distance between points on $\mathcal{B}$ and points in the complement:

$$\sigma_0 : \mathcal{B} \times \mathring{\Omega}_0 \to \mathbb{R},\ (x, w) \mapsto ||x - w||^2.$$

The Maxwell set of $\sigma_0$ is the medial axis of the complement, and we give it a special name.

**Definition 3.3.1.** *For a collection of disjoint compact connected regions $\{\Omega_i\}_{i=1}^{q}$ as above, the* ***linking medial axis****, $M_0$, is the Blum medial axis of the complementary region $\Omega_0$.*

**Remark 3.3.2.** For now, we assume that the linking medial axis extends indefinitely as the complement is unbounded. However, in Chapter 6, we shall develop a finite version of the medial linking structure to be used in applications by assuming that the regions are contained in a bounded region and truncating the linking medial axis in an appropriate manner.

Given $w_i \in \overset{\circ}{\Omega}_i$, distance genericity implies that there is a finite non-empty set $S_i \subset \mathcal{B}_i$ such that $\sigma_i(\cdot, w_i)$ takes on an absolute minimum value $y_i$ at the points in $S_i$. Let $|S_i|$ denote the number of elements in $S_i$. If $|S_i| > 1$, or $S_i = \{x\}$ and $\sigma_i(\cdot, w_i)$ has a degenerate minimum at $x$, then $w_i$ is a point on the Blum medial axis $M_i$ of $\Omega_i$. We refer to $M_i$ as the *internal medial axis of* $\Omega_i$. The points of $S_i$ will be called the *boundary points associated to* $w_i$, and in turn, $w_i$ will be called the *internal medial axis point associated to the points of* $S_i$.

Next, we consider a point $w_0 \in \overset{\circ}{\Omega}_0$, the unbounded component of the complement. Then, by distance genericity, there is a finite non-empty set $S_0 \subset \mathcal{B}$ such that, at the points in $S_0$, $\sigma_0(\cdot, w_0)$ takes on an absolute minimum value $y_0$. Now, $S_0$ may contain points from more than one component $\mathcal{B}_i$. As above, if $|S_0| > 1$, or $S_0 = \{x\}$ and $\sigma_0(\cdot, w_0)$ has a degenerate minimum at $x$, then $w_0$ is a point on the linking medial axis $M_0$. We refer to the points of $S_0$ as the *boundary points associated to* $w_0$, and we say $w_0 \in M_0$ is the *linking medial axis point associated to the points of* $S_0$.

It is important to note that not every point on the boundary of a region will have an associated linking medial axis point. Consider a 2-dimensional example in Figure 3.1. The tangent lines represent the limits of bitangent circles at infinity. The points $x_1, x_2 \in \mathcal{B}_2$ do not have associated linking medial axis points, as the centers of their corresponding bitangent circles are both at infinity. For the same reason, all of the points between $x_1$ and $x_2$ on the side of $\mathcal{B}_2$ furthest from the other regions do not have associated linking medial axis points.

Let $w_0 \in M_0$, and suppose that $w_0$ has $r$ associated boundary points in $S_0 \subset \mathcal{B}$; equivalently, each $x \in S_0$ has $w_0$ as its associated linking medial axis point. Now, each

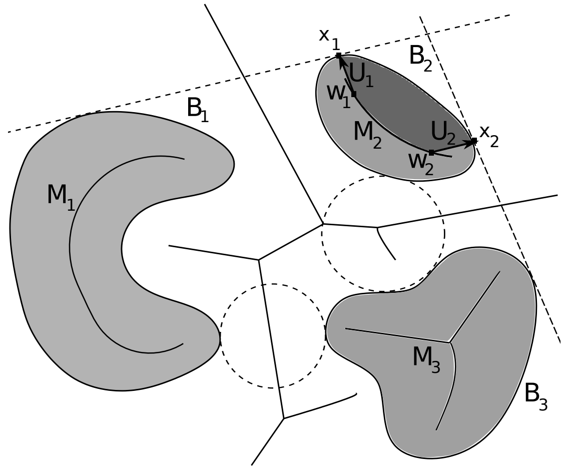


FIGURE 3.1. *Non–linked portions of regions.*
*Examples of bitangent circles in the complement whose centers are linking medial axis points associated to certain boundary points, as well as limits of bitangent circles whose centers are at infinity (i.e., the two bitangent lines). All of the regions must lie on one side of a bitangent line in order for it to correspond to a limit of bitangent circles contained entirely within the complementary region. The darker shading indicates the region between $M_2$ and $\mathcal{B}_2$ which does not have linking medial axis points associated to it.*

point $x$ belongs to some $\mathcal{B}_i$ and is simultaneously associated to a unique internal medial axis point $w_i \in M_i$. See Figure 3.2 for an example. Therefore, each $x$ is "playing dual roles," as it must satisfy multiple conditions as a simultaneous absolute minimum of two different distance functions: $\sigma_0(\cdot, w_0)$ and $\sigma_i(\cdot, w_i)$. This brings us to the definition of various types of *linking*.

**Definition 3.3.3.** *Suppose there are $r$ distinct boundary points associated to $w_0 \in M_0$, namely, $\{x_{j_1}, \ldots, x_{j_{s_j}}\}_{j=1}^{q}$ for $x_{j_i} \in \mathcal{B}_j$ and $\sum_{j=1}^{q} s_j = r$, with possibly $s_j = 0$ for some values of $j$. Suppose $x_{j_i} \in \mathcal{B}_j$ is associated to the internal medial axis point $w_{j_i} \in M_j$. Then the regions $\Omega_1, \ldots, \Omega_q$ are said to be **linked** at $\{w_{j_1}, \ldots, w_{j_{s_j}}\}_{j=1}^{q}$. If $s_j = 0$ for some $j$, then $\Omega_j$ is not included in the list of linked regions.*

*For a given fixed $j$, if $s_i = 0$ for $i \neq j$ and $s_j \neq 0$, we say the region $\Omega_j$ is **self-linked** at $w_{j_1}, \ldots, w_{j_{s_j}}$.*

*Linking may consist entirely of linking between distinct regions (i.e., $s_j = 0$ or $1$ for all $j$), entirely of self-linking (i.e., for only one value of $j$ is $s_j \neq 0$), or it may consist*

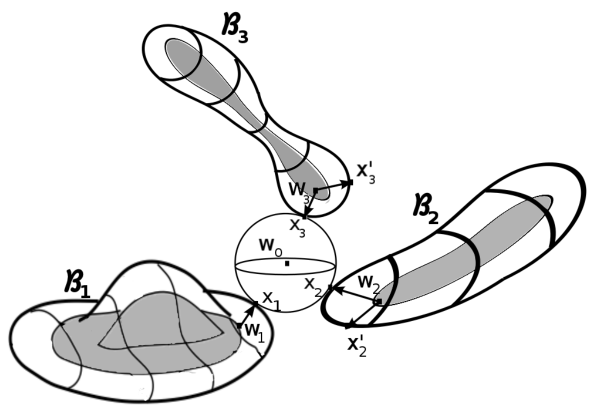


FIGURE 3.2. *Example demonstrating the dual roles of boundary points in* $\mathbb{R}^3$. *Here,* $w_0$ *is the center of a sphere in the complement that is tri-tangent at the surface points* $x_1, x_2$, *and* $x_3$; *thus,* $w_0$ *belongs to a branch curve of the linking medial axis. Furthermore, each point* $x_i \in \mathcal{B}_i, i = 1, 2, 3,$ *has an associated internal medial axis point* $w_i$. *Here,* $w_1$ *is a point on an edge curve of the medial axis of* $\mathcal{B}_1$, *and* $w_i$ *for* $i = 2, 3$ *is the center of a bi-tangent sphere with ordinary tangencies at* $x_i$ *and* $x_i'$.

*of* ***partial linking****, which involves both distinct and self-linking. See Figure 3.3 for examples.*

**Remark 3.3.4.** A couple of remarks are necessary:

*(1)* We refer to $w_0 \in M_0$ in Definition 3.3.3 as an *associated linking point* of any of the $w_{j_i} \in M_j$. Now, $w_{j_i}$ may have other associated linking points corresponding to its other associated boundary points. To distinguish between the associated linking points of $w_{j_i}$, we say $w_0$ is its associated linking point *in the direction of* $U_j$, where $U_j$ is the choice of radial vector at $w_{j_i}$ pointing toward $w_0$.

*(2)* Self-linking includes those cases for which either $\sigma_0(\cdot, w_0)$ has a single degenerate minimum at some $x_i \in \mathcal{B}_i$, or $\sigma_i(\cdot, w_i)$ has an absolute minimum with critical value $y_i$ at a point $x_i \in \mathcal{B}_i$ and at finitely many other points in $\mathcal{B}_i$.

As mentioned earlier, it is possible for points on $M_i$ to be linked to other regions on one side but not the other. For this reason, we shall in fact consider linking on the level of the double $\widetilde{M}_i$ of each medial axis $M_i$, as defined in 2.3.2 in Chapter 2. This enables us to consider both sides of the medial axis simultaneously and distinguish between the type

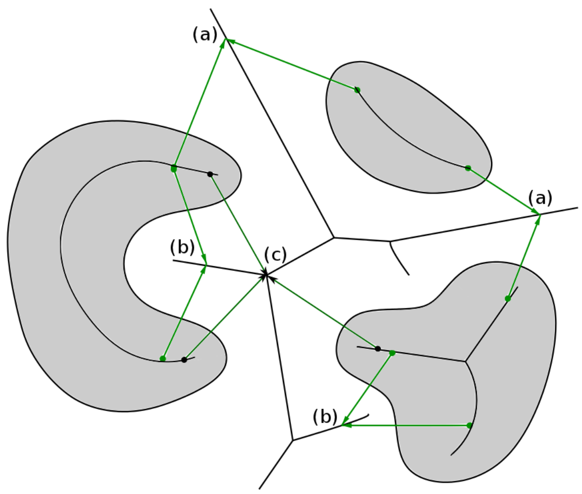


Figure 3.3. *Illustration of (a) linking between distinct regions, (b) self-linking, and (c) partial linking.*

of linking that is exhibited; see Figure 3.4 for an example. It also allows us to determine the portion of the medial axis that consists of non-linked points, as defined below.

**Definition 3.3.5.** *The **non-linked portion of** $\widetilde{M}_i$, which we denote by $\widetilde{M}_i^\infty$, consists of all points $(w, U_i) \in \widetilde{M}_i$ such that $w$ has no associated linking point in the direction of $U_i$. $\widetilde{M}_i^\infty$ may consist of more than one connected component. The **non-linked portion of** $M_i$, denoted $M_i^\infty$, consists of those points $w \in M_i$ with no associated linking points.*

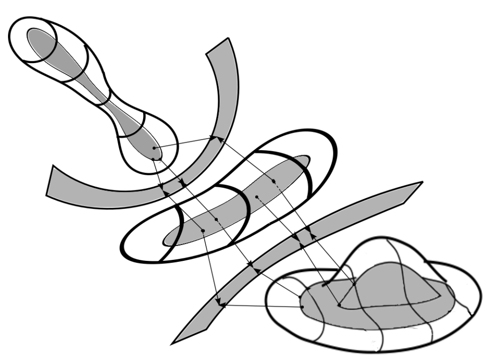

Figure 3.4. *Linking between three regions in $\mathbb{R}^3$.*
*Only a small portion of the linking medial axis is shown for simplicity.*

Let $\mathcal{B}^\infty = \coprod_{i=1}^{q} \mathcal{B}_i^\infty$ denote the *nonlinked portion of* $\mathcal{B}$ consisting of points on $\mathcal{B}$ associated to points in $\coprod_{i=1}^{q} \widetilde{M}_i$. The following proposition establishes the generic properties of the boundary of $\widetilde{M}_i^\infty$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ by establishing the generic properties of $\mathcal{B}^\infty$. Its proof requires Wall's extension of Looijenga's transversality theorem (Theorem 1.4.2 in Chapter 1).

**Proposition 3.3.6.** *For a generic embedding $\phi : X \to \mathbb{R}^2$, the boundary of $\mathcal{B}^\infty$ consists of $A_1^2$ bitangent points. For a generic embedding $\phi : X \to \mathbb{R}^3$, the boundary of $\mathcal{B}^\infty$ consists of smooth curves of $A_1^2$ bitangent points ending at either $A_1^3$ tritangent points or $A_3$ tangent points.*

PROOF. Let $\phi : X \to \mathbb{R}^{n+1}$ for $n = 1$ or 2 be a generic embedding, and let $S^n$ denote the unit $n$–sphere in $\mathbb{R}^{n+1}$. By Theorem 1.4.2, the family of height functions

$$h : (\mathcal{B} \times S^n, (S, w_0)) \to (\mathbb{R}, c), \qquad (x, w) \mapsto w \cdot x \tag{3.1}$$

is a versal unfolding for $S \subset \mathcal{B}$ a finite set of distinct points. Hence, only multigerms of extended $\mathcal{R}^+$–codimension $\leq n$ occur generically as the sphere is $n$–dimensional.

For $n = 1$, the only $\mathcal{R}_e^+$–codimension 1 versally unfolded multigerm is type $A_1^2$. If $S = \{x_1, x_2\} \subset \mathcal{B}$, $h(\cdot, w_0)$ being of type $A_1^2$ at $S$ corresponds geometrically to $w_0$ being orthogonal to $T_{x_i}\mathcal{B}$ for $i = 1, 2$. Generically, the $A_1^2$ set on $\mathcal{B}$ is a set of points bounding curves of nonlinked points of type $A_1$, and the $A_1^2$ points on $\mathcal{B}$ then correspond to points at the boundary of $\widetilde{M}_i^\infty$.

For $n = 2$, the $\mathcal{R}_e^+$–codimension 1 set of $A_1^2$ points now consists of smooth curves on $\mathcal{B}$ corresponding geometrically to a continuum of bitangent planes with all regions lying on one side of them. (The bitangent planes are the limits of bitangent spheres contained entirely inside the complementary region.) In addition, there are two multigerms having extended $\mathcal{R}^+$–codimension 2. First, if $S = \{x_1, x_2, x_3\} \subset \mathcal{B}$, $h(\cdot, w_0)$ could generically be of type $A_1^3$ at $S$, which corresponds geometrically to $w_0$ being orthogonal to $T_{x_i}\mathcal{B}$ for $i = 1, 2, 3$, i.e., it corresponds to a tritangent plane to $\mathcal{B}$ at the points of $S$. These $A_1^3$ tritangent points on $\mathcal{B}$ occur at the end of two $A_1^2$ curves. Second, $h(\cdot, w_0)$ may generically have an $A_3$ singularity at $x_1 \in \mathcal{B}$, and such an $A_3$ point on $\mathcal{B}$ also occurs when a smooth $A_1^2$ curve ends. Therefore, the corresponding boundary of $\widetilde{M}_i^\infty$ also generically consists of smooth curves ending at points. $\square$

## 3.4. Classification of generic medial linking in $\mathbb{R}^2$ and $\mathbb{R}^3$

In this section, we state the classification of the generic local normal forms that a medial linking structure will exhibit in $\mathbb{R}^2$ and $\mathbb{R}^3$. Before doing so, we shall revisit results from Chapter 1 on the singularity theory needed to construct generic normal forms for the families of distance squared functions and their Maxwell sets.

**3.4.1. Singularity theory needed for classification.** Recall from Section 1.2 in Chapter 1 that the ring of multigerms of smooth functions $f : (\mathbb{R}^n, S) \to (\mathbb{R}, 0)$, where $S$ is a set of $r$ distinct points in $\mathbb{R}^n$, is the product of $r$ copies of $\mathfrak{m}_n$. Thus, we may view a multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}, 0)$ as the $r$-tuple of mappings $(f_1, \ldots, f_r)$ with each $f_i : (\mathbb{R}^n, x_i) \to (\mathbb{R}, 0)$ a germ of a smooth function at $x_i \in S \subset \mathbb{R}^n$. Source coordinates may be chosen independently around each $x_i \in S$ so that $x_i$ is locally the origin in each copy of $\mathbb{R}^n$. We shall restrict our attention to families of multigerms satisfying a specific requirement.

Suppose the multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}, 0)$ has an $\mathcal{A}_\beta$ singularity, so that at $x_i \in S$ for $i = 1, \ldots, r$, $f_i$ has an $A_{k_i}$ singularity for some positive integer $k_i$. We say that the family

$$F = (F_1, \ldots, F_r) : (\mathbb{R}^n \times \mathbb{R}^p, (S, 0)) \to (\mathbb{R}, 0)$$

satisfying $F(x, 0) = f(x)$ is of *generic $\mathcal{A}_\beta$ singularity type* if the family is a versal unfolding of the multigerm $f$. We saw in Section 1.2.1 in Chapter 1 that the versality of $F$ implies that it has an algebraic normal form. Recall further that the standard miniversal unfolding of $f$ is the family $F = (F_1, \ldots, F_r)$, where

$$
\begin{aligned}
(3.2) \qquad F_1 &= \sum_{i=1}^{n-1} x_i^2 + x_n^{k_1+1} + \sum_{j=1}^{k_1} u_j x_n^{k_1-j}, \\
F_2 &= \sum_{i=1}^{n-1} x_i^2 + x_n^{k_2+1} + \sum_{j=1}^{k_2} u_{k_1+j} x_n^{k_2-j}, \\
&\vdots \\
F_{r-1} &= \sum_{i=1}^{n-1} x_i^2 + x_n^{k_{r-1}+1} + \sum_{j=1}^{k_{r-1}} u_{(k_1+\ldots+k_{r-2})+j}\, x_n^{k_{r-1}-j}, \\
F_r &= \sum_{i=1}^{n-1} x_i^2 + x_n^{k_r+1} + \sum_{j=1}^{k_r\ -1} u_{(k_1+\ldots+k_{r-1})+j}\, x_n^{k_r-j}.
\end{aligned}
$$

It is important to mention that, for simplicity, we did not distinguish between the different coordinates for each function $F_i$ above; however, each function is defined on a distinct copy of $R^n$, and thus the coordinates are indeed different in each case. Recall from Section 1.2.1 in Chapter 1 that the number of unfolding parameters in its miniversal unfolding equals $\mathcal{R}_e^+$-codim$(\mathcal{A}_\beta)$; that is, $\mathcal{R}_e^+$-codim$(\mathcal{A}_\beta) = k_1 + \ldots + k_r - 1$.

In the situation at hand, for a point $w_i \in M_i,\ i = 1, \ldots, q$, with associated boundary points in the set $S_i \subset \mathcal{B}_i$ and $|S_i| = y_i$, suppose the multigerm

$$\sigma_i(\cdot, w_i) : (\mathcal{B}_i, S_i) \to (\mathbb{R}, y_i)$$

has an $\mathcal{A}_{\beta_i}$ singularity. Similarly, for a point $w_0 \in M_0$ with associated boundary points in the set $S_0 \subset \mathcal{B}$, $|S_0| = r$, suppose the multigerm

$$\sigma_0(\cdot, w_0) : (\mathcal{B}, S_0) \to (\mathbb{R}, y_0)$$

has an $\mathcal{A}_\alpha$ singularity.

**Definition 3.4.1.** *We say that the family*

$$\sigma_i : (\mathcal{B}_i \times \mathbb{R}^{n+1}, (S_i, w_i)) \to (\mathbb{R}, r_i)$$

*is of* ***generic*** $\mathcal{A}_{\beta_i}$ ***singularity type*** *if* $\sigma_i$ *is a versal unfolding of the multigerm* $\sigma_i(\cdot, w_i)$*.*

*Similarly, we say that the family*

$$\sigma_0 : (\mathcal{B} \times \mathbb{R}^{n+1}, (S_0, w_0)) \to (\mathbb{R}, y_0)$$

*is of* ***generic*** $\mathcal{A}_\alpha$ ***singularity type*** *if* $\sigma_0$ *is a versal unfolding of the multigerm* $\sigma_0(\cdot, w_0)$*.*

By the uniqueness theorem for versal unfoldings of multigerms (Theorem 1.2.11 in Chapter 1), $\sigma_i$ (resp., $\sigma_0$) is isomorphic to the unfolding $F_i \times \mathrm{id}_{n+1-p_i}$ (resp., $F_0 \times \mathrm{id}_{n+1-p_0}$), where $F_i$ (resp., $F_0$) is a $p_i$–parameter (resp., $p_0$–parameter) miniversal unfolding of the form given in (3.2).

Recall from Section 1.5.2 in Chapter 1 that the versality theorem ensures that the projection of the stratum of the singular set of $\sigma_i$ corresponding to $\mathcal{A}_{\beta_i}$ singularity type to the parameter space $\mathbb{R}^{n+1}$ will be smooth. Let $\chi_{\mathcal{A}_{\beta_i}} \subset M_i$ denote this projection. In addition, let $\Sigma_{\mathcal{A}_{\beta_i}} \subset \mathcal{B}_i$ denote the projection to $\mathcal{B}_i$ of the stratum for $\mathcal{A}_{\beta_i}$ singularity type. The $\Sigma_{\mathcal{A}_{\beta_i}}$ stratum on $\mathcal{B}_i$ is the image under the radial flow of the $\chi_{\mathcal{A}_{\beta_i}}$ stratum on $M_i$. Since the radial flow is smooth on the strata of the medial axis [**14**], this ensures that $\Sigma_{\mathcal{A}_{\beta_i}}$ is smooth. Furthermore, let $\Sigma_{\mathcal{A}_\alpha} \subset \mathcal{B}$ denote the projection to $\mathcal{B}$ of the stratum of the singular set of $\sigma_0$ corresponding to $\mathcal{A}_\alpha$ singularity type, and let $\chi_{\mathcal{A}_\alpha}$ denote its projection to $M_0$. Linking occurs when, for some $i$, there is at least one point $x_i \in \mathcal{B}_i$ such that $x_i$ is a singular point of both $\sigma_i(\cdot, w_i)$ and $\sigma_0(\cdot, w_0)$. When linking occurs, the singular sets $\Sigma_{\mathcal{A}_{\beta_i}}$ and $\Sigma_{\mathcal{A}_\alpha}$ intersect on the boundary $\mathcal{B}_i$. As will be explained in Chapter 5, the classification of generic linking depends on the transverse intersection of these singular sets on the boundaries.

**3.4.2. Generic linking classification theorems.** The classification for generic medial linking is expressed using a normal form for the families of versal unfoldings in the previous section.

**Definition 3.4.2.** *Let* $S_0 = \{x_{j_1}, \ldots, x_{j_{s_j}}\}_{j=1}^q \subset \mathcal{B}$ *and* $|S_0| = \sum_{j=1}^{q} s_j = r$*, with possibly* $s_j = 0$ *for some values of* $j$*. Let* $\sigma_0 : (\mathcal{B} \times \mathbb{R}^{n+1}, (S_0, w_0)) \to (\mathbb{R}, y_0)$ *be of generic* $\mathcal{A}_\alpha$

*singularity type. Also, let* $S_{j_i} \subset \mathcal{B}_j$ *be a finite set of distinct points with* $x_{j_i} \in S_{j_i} \cap S_0$*, and let* $\sigma_{j_i} : (\mathcal{B}_j \times \mathbb{R}^{n+1}, (S_{j_i}, w_{j_i})) \to (\mathbb{R}, y_{j_i})$ *be of generic* $\mathcal{A}_{\beta_{j_i}}$ *singularity type. For* $j = 1, \ldots, q$ *and* $i = 1, \ldots, s_j$*, a* ***generic linking configuration*** *consists of the set of* $\{\sigma_{j_i}\}$ *and* $\sigma_0$ *with the requirement that the singular sets* $\Sigma_{\mathcal{A}_\alpha} \subset \mathcal{B}$ *and* $\Sigma_{\mathcal{A}_{\beta_{j_i}}} \subset \mathcal{B}_j$ *intersect transversely in* $\mathcal{B}$*.*

*We denote the* ***normal form for a generic linking configuration*** *by*

$$(\mathcal{A}_\alpha : \mathcal{A}_{\beta_1}, \ldots, \mathcal{A}_{\beta_r}) \tag{3.3}$$

*or, letting* $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_r\}$*, by*

$$(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}}). \tag{3.4}$$

In any dimension, there are four basic categories to which linking configurations belong:

**Type 0 Linking**: Smooth strata on internal medial axes linked at a smooth stratum of the linking medial axis.

**Type I Linking**: Singular stratum on one internal medial axis linked to smooth/singular stratum on other internal medial axis at a smooth stratum of the linking medial axis.

**Type II Linking**: Smooth strata on internal medial axes linked at a singular stratum of the linking medial axis.

**Type III Linking**: Singular stratum on one internal medial axis linked to smooth/singular strata on other internal medial axes at a singular stratum of the linking medial axis.

Next, we state the generic medial linking and medial self-linking classification theorems for dimensions 2 and 3 using the notation in (3.3). The proofs of the theorems will occupy the next two chapters.

**Theorem 3.4.3** (**Classification of generic linking in** $\mathbb{R}^2$)**.** *For a residual set of embeddings of simple disjoint closed curves in* $\mathbb{R}^2$ *whose interiors bound disjoint regions, there are 5 generic linking types, given in Table 3.1. Of these, only the first 4 can occur for linking between distinct regions, while all 5 can occur for self-linking.*

TABLE 3.1. Generic linking for medial axes in $\mathbb{R}^2$.

| Linking type | Configuration |
|---|---|
| 0 | $(A_1^2 : A_1^2, A_1^2)$ |
| I | $(A_1^2 : A_1^2, A_1^3)$<br>$(A_1^2 : A_1^2, A_3)$ |
| II | $(A_1^3 : A_1^2, A_1^2, A_1^2)$ |
| II | $(A_3 : A_1^2)$ |

**Theorem 3.4.4 (Classification of generic linking in $\mathbb{R}^3$).** *For a residual set of embeddings of smooth, disjoint, compact, connected surfaces in $\mathbb{R}^3$ whose interiors bound disjoint regions, there are 17 generic linking types, given in Table 3.2. Of these, only the first 13 can occur for linking between distinct regions, but all 17 can occur for self-linking.*

TABLE 3.2. Generic linking for medial axes in $\mathbb{R}^3$.

| Linking type | Configuration |
|---|---|
| 0 | $(A_1^2 : A_1^2, A_1^2)$ |
| I | $(A_1^2 : A_1^2, A_1^3)$<br>$(A_1^2 : A_1^2, A_3)$<br>$(A_1^2 : A_1^3, A_1^3)$<br>$(A_1^2 : A_1^3, A_3)$<br>$(A_1^2 : A_3, A_3)$<br>$(A_1^2 : A_1^2, A_1^4)$<br>$(A_1^2 : A_1^2, A_1A_3)$<br>(2 cases) |
| II | $(A_1^3 : A_1^2, A_1^2, A_1^2)$<br>$(A_1^4 : A_1^2, A_1^2, A_1^2, A_1^2)$<br>$(A_1A_3 : A_1^2, A_1^2)$ |
| III | $(A_1^3 : A_1^2, A_1^2, A_1^3)$<br>$(A_1^3 : A_1^2, A_1^2, A_3)$ |
| II | $(A_3 : A_1^2)$ |
| III | $(A_3 : A_1^3)$<br>$(A_3 : A_3)$ |

**Remark 3.4.5.** There are two distinct cases with generic linking type $(A_1^2 : A_1^2, A_1A_3)$. Namely, if the $A_1^2$ multigerm corresponding to the linking medial axis point is taken at $x_i \in \mathcal{B}_i$ and $x_j \in \mathcal{B}_j$, and $\sigma_j(\cdot, w_j)$ is of generic singularity type $A_1A_3$ with $w_j \in M_j$ the

internal medial axis point associated to $x_j$, then the two cases depend on whether $x_j$ is an $A_1$ singular point or an $A_3$ singular point.

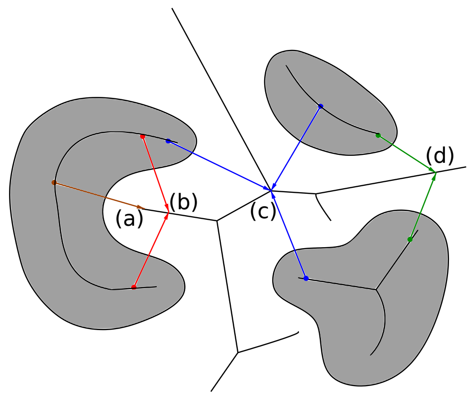


FIGURE 3.5. *Example illustrating generic linking possibilities, including self-linking, in* $\mathbb{R}^2$.

**Example 3.4.6.** Consider Figure 3.5. The points labeled (a) and (b) provide examples of $(A_1^2 : A_1^2, A_1^2)$ and $(A_3 : A_1^2)$ self-linking, respectively. The point (c) illustrates the generic linking type $(A_1^3 : A_1^2, A_1^2, A_1^2)$. The three points on the region boundaries that correspond to (c) are associated simultaneously to the branch point on the linking medial axis and to top-dimensional strata on the regions' medial axes. The point (d), which illustrates the generic linking type $(A_1^2 : A_3, A_1^2)$, belongs to a smooth curve of the linking medial axis and links a smooth stratum on one internal medial axis to an edge point on the other.

### 3.5. Details of the labeled stratification refinements

The classification of generic linking naturally leads to refinements of the usual Whitney stratifications of $\mathcal{B} = \coprod_{i=1}^{q} \mathcal{B}_i$ and $M = \coprod_{i=0}^{q} M_i$. These refinements will arise from the Maxwell set related singular behavior of the versal unfolding $\sigma_0$ in addition to that of each versal unfolding $\sigma_i, i = 1, \ldots, q$. In this section, we shall define the stratification refinements and examine their important features and properties. We remark that, for the medial axes, the local topological structure of the ordinary stratification identifies each multigerm type for a point in a stratum. This will no longer be true for the refinement. Hence, we shall use "labeled refinements."

We begin by explaining the process by which the stratifications are refined, which shows how the stratifications on $\mathcal{B}$ and $M$ are related. This process requires us to study linking on each side of the medial axis separately. This means that for any point $w_i \in M_i \setminus M_i^\infty$, we consider only one of its associated linking points in $M_0$ at a time.

In Section 3.3, we introduced the correspondence between points in $\mathcal{B}_i$ and points in $M_i$, as well as the correspondence between linking medial axis points in $M_0$ and points in $\mathcal{B}$. The versal unfoldings $\sigma_i$ and $\sigma_0$ provide these correspondences. From them, we may determine for any point $x_i$ in a given stratum $\Sigma_{\mathcal{A}_{\beta_i}}$ on $\mathcal{B}_i$ (arising from the singular set of $\sigma_i$) its associated point $w_i$ in $M_i$, which belongs to the stratum $\chi_{\mathcal{A}_{\beta_i}}$ in $M_i$. Moreover, we may determine the point in $M_0$ associated to $x_i$, which belongs to some stratum of the usual stratification of $M_0$. The set of all points in $M_0$ corresponding to the $\Sigma_{\mathcal{A}_{\beta_i}}$ stratum on $\mathcal{B}_i$ are labeled with the kind of singular behavior they represent for $\sigma_i$. We refer to this set as the *image of the* $\Sigma_{\mathcal{A}_{\beta_i}}$ *stratum on* $\mathcal{B}_i$ *in* $M_0$. (We may also refer to it as the *image of the* $\chi_{\mathcal{A}_{\beta_i}}$ *stratum on* $M_i$ *in* $M_0$.) Furthermore, each of these points in $M_0$ corresponds to one or more distinct points on other boundaries; we refer to the set of all corresponding points in $\mathcal{B}_j$ as the *image of the* $\Sigma_{\mathcal{A}_{\beta_i}}$ *stratum on* $\mathcal{B}_i$ *in* $\mathcal{B}_j$ and label them accordingly. Finally, any point in $\mathcal{B}_j$ that belongs to this image is associated to a point in $M_j$, which enables us to determine the *image of the* $\Sigma_{\mathcal{A}_{\beta_i}}$ *stratum on* $\mathcal{B}_i$ *in* $M_j$.

Likewise, any point in the stratum $\Sigma_{\mathcal{A}_\alpha}$ on $\mathcal{B}_i$ (arising from the singular set of $\sigma_0$) has an associated point in $M_0$; the set of these associated points form the $\chi_{\mathcal{A}_\alpha}$ stratum in $M_0$. Using the process just described, we may determine the *images of the* $\Sigma_{\mathcal{A}_\alpha}$ *stratum on* $\mathcal{B}_i$ *in* $M_i$, $\mathcal{B}_j$, *and* $M_j$. See Figures 3.6 and 3.7 for an illustration.

With this explanation in mind, we give the following intrinsic definition of a stratum on $\mathcal{B}$ arising from generic linking type.

**Definition 3.5.1.** *Let* $\mathcal{B} = \coprod_{i=1}^{q} \mathcal{B}_i$ *and let* $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$ *be a generic linking configuration with* $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_r\}$ *as in Definition 3.4.2. A point* $x_i \in \mathcal{B}_i$ *belongs to the* ***stratum on*** $\mathcal{B}$ ***corresponding to the configuration*** $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$, *denoted* $(\Sigma_\alpha : \Sigma_{\boldsymbol{\beta}})^{\mathcal{B}}$, *provided the following conditions hold:*

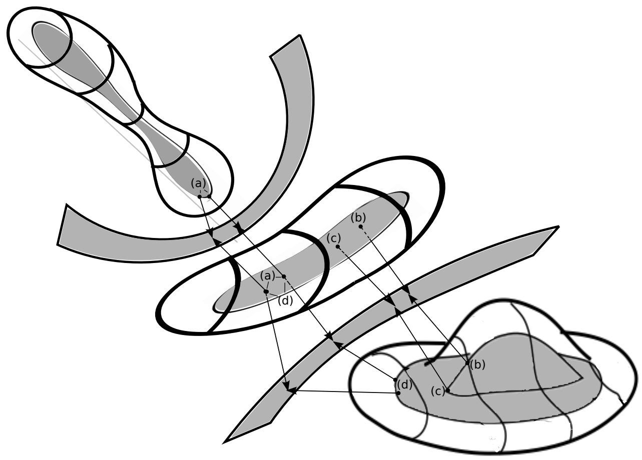


FIGURE 3.6. *Stratification refinement example.*
*Revisiting Figure 3.4 in order to determine the two sides of the refined stratification of the medial axis of the middle region.*

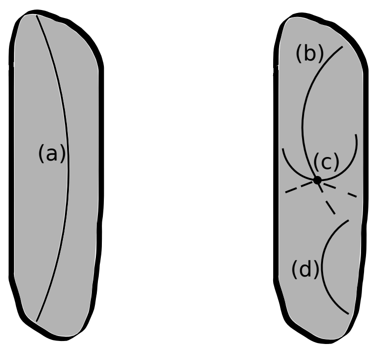


FIGURE 3.7. *More on the refinement example.*
*On the left, the refinement due to linking with the leftmost region; on the right, a portion of the refinement from linking with the far right region.*

- $x_i \in \Sigma_{\mathcal{A}_\alpha}$ *on* $\mathcal{B}$*;*
- $x_i \in \Sigma_{\mathcal{A}_{\beta_i}}$ *on* $\mathcal{B}_i$*;*
- *There exists a set of* $r$ *points* $S_0 \subset \mathcal{B}$ *with* $x_i \in S_0$ *such that, for any other point* $x_j \in S_0$ *with* $x_j \in \mathcal{B}_j$*,* $x_j \in \Sigma_{\mathcal{A}_\alpha}$ *on* $\mathcal{B}$ *and* $x_j \in \Sigma_{\mathcal{A}_{\beta_j}}$ *on* $\mathcal{B}_j$*;*
- *For every* $j = 1, \ldots, r$*, the* $\Sigma_{\mathcal{A}_\alpha}$ *stratum, the* $\Sigma_{\mathcal{A}_{\beta_i}}$ *stratum, and the images in* $\mathcal{B}_i$ *of the* $\Sigma_{\mathcal{A}_{\beta_j}}$ *strata are transverse in* $\mathcal{B}_i$*.*

**Definition 3.5.2.** *Let* $\mathscr{S}^{\mathcal{B}}$ *denote the collection of all strata of the form in Definition 3.5.1, and let* $\mathscr{S}^{\mathcal{B}_i}$ *denote the restriction of* $\mathscr{S}^{\mathcal{B}}$ *to* $\mathcal{B}_i$ *for* $i = 1, \ldots, q$*.*

The next theorem establishes a genericity result regarding the stratification in Definition 3.5.2. We refer the reader to Section 3.2 to recall the precise meaning of a generic embedding $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$.

**Theorem 3.5.3.** *For a generic $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ with $n \leq 6$, $\mathscr{S}^{\mathcal{B}}$ forms a Whitney stratification of $\mathcal{B}$.*

Theorem 3.5.3 will follow from the combination of Theorem 4.3.1 in Chapter 4 and the work in Chapter 5. Briefly, the fact that $\mathscr{S}^{\mathcal{B}}$ is a Whitney stratification will be a consequence of the fact given in Section 1.5.2 in Chapter 1 that the canonical stratification of jet space by singularity type is a Whitney stratification. The transversality results of the next two chapters will imply that the pull-backs of the stratifications involved in a generic linking configuration to $\mathcal{B}$ will be transverse, and will therefore form a Whitney stratification of $\mathcal{B}$.

Next, we present the corresponding results for the stratifications of the medial axes. In Section 3.3, we explained that linking is best understood on the level of the double $\widetilde{M}_i$ for each $i = 1, \ldots, q$. This is because its topological structure enables us to study linking with other regions that is exhibited on each side of the medial axis separately. On the other hand, we shall always consider strata on $M_0$ rather than $\widetilde{M}_0$ because linking simultaneously involves both sides of $M_0$.

**Definition 3.5.4.** *Let $\widetilde{M} = \coprod_{i=1}^{q} \widetilde{M}_i$ and let $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$ be a generic linking configuration with $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_r\}$ as in Definition 3.4.2.*

*(a) A point $(w_i, U_i) \in \widetilde{M}_i$ belongs to the* ***stratum on $\widetilde{M}$ corresponding to the configuration*** *$(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$, denoted $(\chi_\alpha : \chi_{\boldsymbol{\beta}})^{\widetilde{M}}$, provided the following hold:*

- *$w_i \in \chi_{\mathcal{A}_{\beta_i}}$ on $M_i$;*
- *For $j = 1, \ldots, r$, there exist points $x_j \in \mathcal{B}_j$, $w_j \in M_j$, and $w_0 \in M_0$ such that for every $j$, the multigerms $\sigma_0(\cdot, w_0)$ and $\sigma_j(\cdot, w_j)$ are singular at the point $x_j$, with $w_0 \in \chi_{\mathcal{A}_\alpha}$ on $M_0$ and $w_j \in \chi_{\mathcal{A}_{\beta_j}}$ on $M_j$;*
- *The images in $M_i$ of the collection of $\chi_{\mathcal{A}_\alpha}$ and $\chi_{\mathcal{A}_{\beta_j}}$ strata are transverse.*

*(b) A point $w_0 \in M_0$ belongs to the* ***stratum on*** $M_0$ ***corresponding to the configuration*** $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$*, denoted* $(\chi_\alpha : \chi_{\boldsymbol{\beta}})^{M_0}$*, provided the following hold:*

- *$w_0 \in \chi_{\mathcal{A}_\alpha}$ on $M_0$;*
- *There exist points $w_i \in M_i$ and $x_i \in \mathcal{B}_i$, $i = 1, \ldots, r$, such that for every $i$, the multigerms $\sigma_0(\cdot, w_0)$ and $\sigma_i(\cdot, w_i)$ are singular at the point $x_i$ and $w_i \in \chi_{\mathcal{A}_{\beta_i}}$ on $M_i$;*
- *The images in $M_0$ of the collection of $\chi_{\mathcal{A}_\alpha}$ and $\chi_{\mathcal{A}_{\beta_j}}$ strata are transverse.*

**Definition 3.5.5.** *Let $\mathscr{S}^{\widetilde{M}}$ denote the collection of all refined strata of $\widetilde{M}$ of the form in Definition 3.5.4, and let $\mathscr{S}^{\widetilde{M}_i}$ denote the restriction of $\mathscr{S}^{\widetilde{M}}$ to $\widetilde{M}_i$ for $i = 1, \ldots, q$.*

*Let $\mathscr{S}^{M_0}$ denote the collection of refined strata of $M_0$ of the form in Definition 3.5.4.*

**Theorem 3.5.6.** *For a generic $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ with $n \leq 6$, $\mathscr{S}^{\widetilde{M}}$ is a refinement of the stratification of $\widetilde{M}$ and $\mathscr{S}^{M_0}$ is a refinement of the stratification of $M_0$. For $n = 1, 2$, $\mathscr{S}^{M_0}$ is a Whitney stratification.*

The stratifications in Theorems 3.5.3 and 3.5.6 are called the **stratifications by linking type of** $\mathcal{B}$, $\widetilde{M}$, and $M_0$. Although $\mathscr{S}^{\widetilde{M}}$ will not be a Whitney stratification since $\widetilde{M}$ is not embedded in some smooth ambient manifold, the stratification $\mathscr{S}^{\mathcal{B}}$ on $\mathcal{B}$ — which is Whitney — may be viewed as a realization of the stratification on $\widetilde{M}$. For each $i = 1, \ldots, q$, the boundary of $M_i^\infty$ (resp., $\widetilde{M}_i^\infty$) will be included as part of the labeled refined stratification $\mathscr{S}^{M_i}$ (resp., $\mathscr{S}^{\widetilde{M}_i}$). However, due to the absence of linking, the stratification within $M_i^\infty$ ($\widetilde{M}_i^\infty$) will not be refined, so it will retain the original stratification of $M_i$ ($\widetilde{M}_i$).

Generically, the stratifications given in Theorems 3.5.3 and 3.5.6 possess certain properties.

**Theorem 3.5.7.** *For a generic $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ with $n \leq 6$, the stratifications $\mathscr{S}^{\mathcal{B}}$ of $\mathcal{B}$, $\mathscr{S}^{\widetilde{M}}$ of $\widetilde{M}$, and $\mathscr{S}^{M_0}$ of $M_0$ have the following properties:*

*(1) The number of generically appearing stratum types is finite.*

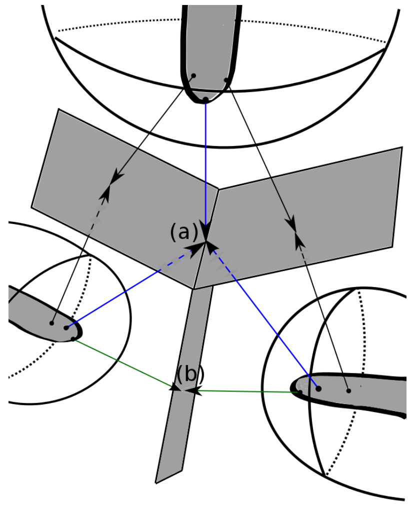


FIGURE 3.8. *Example of generic linking between three distinct regions in* $\mathbb{R}^3$. *The linking that occurs at (a) is an example of an* $(A_1^3 : A_3, A_1^2, A_1^2)$ *configuration, while the* $(A_1^2 : A_3, A_3)$ *configuration is exhibited at (b).*

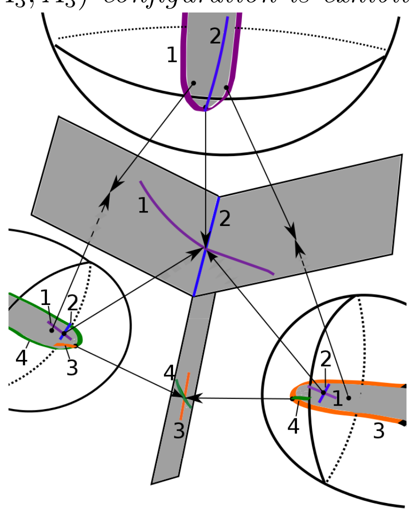


FIGURE 3.9. *A portion of the refined stratification for Figure 3.8. The original strata on each of the medial axes are further decomposed in several places by transversely intersecting curves and their intersection points.*

*(2) The codimension in* $\mathcal{B}$ *of the stratum* $(\Sigma_\alpha : \Sigma_{\boldsymbol{\beta}})^{\mathcal{B}} \in \mathscr{S}^{\mathcal{B}}$ *representing the generic linking configuration* $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$ *is given by the formula*

$$codim_{\mathcal{B}}((\Sigma_\alpha : \Sigma_{\boldsymbol{\beta}})^{\mathcal{B}}) = \mathcal{R}_e^+\text{-}codim(\mathcal{A}_\alpha) + \sum_{i=1}^{r} \mathcal{R}_e^+\text{-}codim(\mathcal{A}_{\beta_i}) - (r+1). \tag{3.5}$$

*(3) Therefore, the codimension in $\mathbb{R}^{n+1}$ of both the stratum $(\chi_\alpha : \chi_{\boldsymbol{\beta}})^{\widetilde{M}} \in \mathscr{S}^{\widetilde{M}}$ and the stratum $(\chi_\alpha : \chi_{\boldsymbol{\beta}})^{M_0} \in \mathscr{S}^{M_0}$ corresponding to $(\mathcal{A}_\alpha : \mathcal{A}_{\beta_1}, \ldots, \mathcal{A}_{\beta_r})$ is given by*

$$codim_{\mathbb{R}^{n+1}}(\chi_\alpha : \chi_{\boldsymbol{\beta}}) = \mathcal{R}_e^+ codim(\mathcal{A}_\alpha) + \sum_{i=1}^{r} \mathcal{R}_e^+ codim(\mathcal{A}_{\beta_i}) - r. \tag{3.6}$$

The fact that only a finite number of stratum types occur generically follows from Mather's classification in Chapter 1; for $n \leq 6$, the dimension is below the dimension when moduli are introduced (i.e., $n = 7$). The remainder of Theorem 3.5.7 is a consequence of a transversality theorem for "multi-distance" functions that will be proven in Chapter 4.

## 3.6. Linking vector fields

In this section, we extend the multivalued radial vector field $U_i = r_i \mathbf{u}_i$ on each medial axis $M_i$ to another multivalued vector field, the *linking vector field*, defined on $M_i$. This vector field will relate $M_i$ to the other medial axes of a collection of regions and to the medial axis of the complement, $M_0$. The new vector field will be defined on $M_i \setminus M_i^\infty$, and will be multivalued with one value at each point $(w, U_i) \in \widetilde{M}_i$ for every associated linking point of $(w, U_i)$. We begin by defining a single-valued linking vector field on the double $\widetilde{M}_i \setminus \widetilde{M}_i^\infty$, which descends to the multivalued linking field on $M_i \setminus M_i^\infty$.

**Definition 3.6.1.** *The **linking vector field** on $\widetilde{M}_i \setminus \widetilde{M}_i^\infty$,*

$$\widetilde{L}_i : \widetilde{M}_i \setminus \widetilde{M}_i^\infty \to \mathbb{R}^{n+1},$$

*associates to a point $(w, U_i)$ the vector that points from $w$ to its associated linking point on $M_0$ in the direction of $U_i$, as defined in Definition 3.3.3. We also define the **linking function***

$$\widetilde{\ell}_i : \widetilde{M}_i \setminus \widetilde{M}_i^\infty \to \mathbb{R}^+, \ (w, U_i) \mapsto ||\widetilde{L}_i(w, U_i)||.$$

*The linking vector field $\widetilde{L}_i$ and linking function $\widetilde{\ell}_i$ descend to a multivalued linking vector field*

$$L_i : M_i \setminus M_i^\infty \to \mathbb{R}^{n+1}$$

*and multivalued linking function*

$$\ell_i : M_i \setminus M_i^\infty \to \mathbb{R}^+.$$

*There is the relation* $L_i(w) = \ell_i(w)\boldsymbol{u}_i(w)$*, where for each choice of unit radial vector* $\boldsymbol{u}_i$ *at* $w \in M_i \setminus M_i^\infty$*,* $L_i(w)$ *is the unique linking vector which is a multiple of* $\boldsymbol{u}_i$*.*

We may rephrase the definitions of linking and self-linking in terms of these vector fields.

**Definition 3.6.2.** *(a) A medial axis* $M_i$ *exhibits* ***linking*** *with a medial axis* $M_j$ *at points* $y \in M_i$ *and* $w \in M_j$ *if there are linking vectors* $L_i(y)$ *and* $L_j(w)$ *satisfying*

$$y + L_i(y) = w + L_j(w).$$

*(b) A medial axis* $M_i$ *exhibits* ***self-linking*** *at points* $y, w \in M_i$ *if there exist linking vectors* $L_i(y)$ *and* $L_i(w)$ *satisfying*

$$y + L_i(y) = w + L_j(w).$$

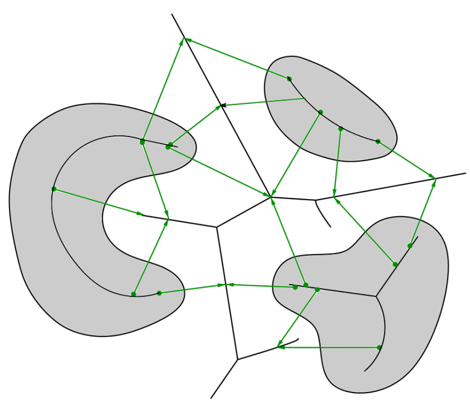

FIGURE 3.10. *A collection of linking vector fields in* $\mathbb{R}^2$*.*

The multivalued linking functions have the following generic properties.

**Lemma 3.6.3.** *For a generic* $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ *with* $n \leq 6$*, the linking function* $\ell_i : M_i \setminus M_i^\infty \to \mathbb{R}^+$ *satisfies the following properties:*

*(1)* $\ell_i(w) > r_i(w)$ *for all* $w \in M_i$.

*(2) If* $y \in M_i$ *and* $w \in M_j$ *are linked, the associated radial and linking functions satisfy the relation*

$$\ell_i(y) - r_i(y) = \ell_j(w) - r_j(w). \tag{3.7}$$

*(3)* $\ell_i$ *is a continuous multivalued function, i.e.,* $\widetilde{\ell}_i$ *as a function on* $\widetilde{M}_i$ *is continuous.*

*(4) In addition,* $\widetilde{\ell}_i|\chi_{ij}$ *is smooth on every stratum* $\chi_{ij}$ *(for all* $j$ *in an index set* $\mathcal{I}$*) in the refined stratification* $\mathscr{S}^{\widetilde{M}_i}$.

Proof. Property 1 is immediate by definition. Next, for linked points $y \in M_i$ and $w \in M_j$, the linking vectors $L_i(y)$ and $L_j(w)$ extend beyond $\mathcal{B}_i$ and $\mathcal{B}_j$ in directions normal to the boundaries (as the linking vectors are in the directions of radial vectors) and meet in the complement. Thus, Property 2 holds since the values in (3.7) are the lengths of radial vectors on $M_0$ that are based at the point $y + L_i(y) = w + L_j(w)$.

To prove Property 3, we show that $\widetilde{\ell}_i$ is continuous which, by definition, means that the multivalued linking function $\ell_i$ is also continuous. In fact, we shall show that the linking vector field $\widetilde{L}_i$ is continuous, which ensures that $\widetilde{\ell}_i$ is continuous.

Let $(w_0, U_i) \in \widetilde{M}_i \setminus \widetilde{M}_i^{\infty}$. By the proof of Theorem 5.1 in [**14**], we know that the radial map

$$\widetilde{\psi}_1^i : \widetilde{M}_i \to \mathcal{B}_i, \ (w, U_i) \mapsto w + U_i(w)$$

is a local homeomorphism from a neighborhood $W$ of $(w_0, U_i) \in \widetilde{M}_i$ to a neighborhood $V$ of $x := w_0 + U_i(w_0) \in \mathcal{B}_i$. In addition, let $w_0' = w_0 + L_i(w_0)$ be the associated linking point of $w_0$ in $M_0$ in the direction of $U_i$; then $x = w_0' + U_0(w_0')$, as well. The result from [**14**] also establishes that the radial map

$$\widetilde{\psi}_1^0 : \widetilde{M}_0 \to \mathcal{B}, \ (w', U_0) \mapsto w' + U_0(w')$$

is a local homeomorphism from a neighborhood $W'$ of $(w_0', U_0) \in \widetilde{M}_0$ to a neighborhood $V'$ of $x \in \mathcal{B}_i$. So, let $V_0 = V \cap V'$, and restrict the radial maps to neighborhoods $W_0$ and

$W_0'$ so that $\widetilde{\psi}_1^i : W_0 \to V_0$ and $\widetilde{\psi}_1^0 : W_0' \to V_0$ are homeomorphisms. It follows that

$$(\widetilde{\psi}_1^0)^{-1} \circ \widetilde{\psi}_1^i : W_0 \to W_0'$$

is also a homeomorphism; in particular, it is continuous. In addition, the natural projection map $\pi : \widetilde{M}_0 \to M_0$ is continuous, as is its restriction to $W_0'$; hence, the composition

$$\pi|W_0' \circ (\widetilde{\psi}_1^0)^{-1} \circ \widetilde{\psi}_1^i : W_0 \to M_0 \tag{3.8}$$

is continuous. Furthermore, by associating to any point $(w, U_i) \in W_0$ the vector that points from $w$ to its image in $M_0$ under the map in (3.8), one obtains the linking vector field $\widetilde{L}_i : W_0 \subset \widetilde{M}_i \setminus \widetilde{M}_i^\infty \to \mathbb{R}^{n+1}$. Therefore, $\widetilde{L}_i$ and $\widetilde{\ell}_i$ are also continuous.

For Property 4, we show that $\widetilde{L}_i$ (and thus $\widetilde{\ell}_i$) is smooth on each stratum in the refined stratification $\mathscr{S}^{\widetilde{M}_i}$ of $\widetilde{M}_i$. Let $(w_0, U_i) \in \widetilde{M}_i \setminus \widetilde{M}_i^\infty$, and let $w_0' := w_0 + L_i(w_0) \in M_0$ be the associated linking point of $w_0$ in the direction of $U_i$, with possibly at least one point $(y_0, U_j) \in \widetilde{M}_j \setminus \widetilde{M}_j^\infty$ for some $j$ such that $w_0' = y_0 + L_j(y_0)$ is the associated linking point of $y_0$ in the direction of $U_j$. If $w_0 \in (M_i)_{\mathrm{reg}}$, let $W$ be a neighborhood of $w_0$. If $w_0 \in (M_i)_{\mathrm{sing}}$, choose a smooth value of $U_i$ on a neighborhood $W$ of $w_0$ within a local manifold component or local edge component. By Proposition 4.1, Corollary 4.3, and Proposition 4.4 in [**14**], the radial map $\psi_1^i : W \to \mathcal{B}_i$ is a local diffeomorphism from $W$ to a neighborhood $V$ of $x := w_0 + U_i(w_0) \in \mathcal{B}_i$.

Similarly, $\psi_1^j : M_j \to \mathcal{B}_j$ is a local diffeomorphism from a neighborhood $\mathcal{O}$ of $y_0 \in M_j$ to a neighborhood $\mathcal{U}$ of $x' := y_0 + U_j(y_0) \in \mathcal{B}_j$. In addition, the radial map $\psi_1^0 : M_0 \to \mathcal{B}$ is a local diffeomorphism from a neighborhood $W'$ of $w_0' \in M_0$ to a neighborhood $V'$ of $x \in \mathcal{B}_i$, and from a neighborhood $\mathcal{O}'$ of $w_0'$ to a neighborhood $\mathcal{U}'$ of $x' \in \mathcal{B}_j$ for smooth choices of the vector field $U_0$ on these neighborhoods.

Thus, let $W_0' = W' \cap \mathcal{O}'$, and restrict to neighborhoods $W_0$ of $w_0 \in M_i$ and $V_0$ of $x \in \mathcal{B}_i$ so that $\psi_1^i : W_0 \to V_0$ and $\psi_1^0 : W_0' \to V_0$ are diffeomorphisms. Therefore,

$$(\psi_1^0)^{-1} \circ \psi_1^i : W_0 \to W_0' \tag{3.9}$$

is also a diffeomorphism, and by associating to any point $w \in W_0$ the vector pointing from $w$ to its image in $M_0$, one obtains the linking vector field on $W_0$. $\square$

### 3.7. Definition of the Blum medial linking structure

Now, we have all of the necessary ingredients to define the *Blum medial linking structure* for a set of compact disjoint regions in $\mathbb{R}^{n+1}$.

**Definition 3.7.1.** *Let $\{\Omega_i\}_{i=1}^q$ be a collection of disjoint regions in $\mathbb{R}^{n+1}$ with smooth boundaries $\partial\Omega_i = \mathcal{B}_i$ defined by a generic embedding $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ for $n \leq 6$. The **Blum medial linking structure** associated to $\{\Omega_i\}_{i=1}^q$ consists of the following:*

*(1) the set of $q$ Blum medial axes $M_i$ for each $\Omega_i$ exhibiting the generic local structure given in Theorem 1.4.4 and multivalued radial vector fields $U_i = r_i\boldsymbol{u}_i$ defined on the $M_i$;*

*(2) the collection of multivalued linking functions $\{\ell_i : M_i \setminus M_i^\infty \to \mathbb{R}^+\}_{i=1}^q$ and multivalued linking vector fields $\{L_i = \ell_i\boldsymbol{u}_i\}_{i=1}^q$ defined on each $M_i \setminus M_i^\infty$ satisfying the properties in Lemma 3.6.3;*

*(3) a linking stratification $\mathscr{S}^{M_0}$ in $\mathbb{R}^{n+1} \setminus \bigcup_{i=1}^q \Omega_i$;*

*(4) a Whitney stratification $\mathscr{S}^{\mathcal{B}_i}$ of each $\mathcal{B}_i$ by linking type; and*

*(5) the collection of labeled refinements $\{\mathscr{S}^{\widetilde{M}_i}\}_{i=1}^q$ of the stratifications of the $\widetilde{M}_i$'s*

*such that the stratifications satisfy the conclusions of Theorems 3.5.3, 3.5.6, and 3.5.7.*

Upon proving the transversality theorem in Chapter 4 and the generic linking classification results for dimensions $n \leq 6$ in Chapter 5, we obtain the following existence theorem.

**Theorem 3.7.2.** *For $n \leq 6$, a generic embedding $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ has a Blum medial linking structure. For $n = 1$ or $2$, the linking structure has the local forms given by the classification in Theorems 3.4.3 and 3.4.4.*

CHAPTER 4

# A Transversality Theorem for Multi–Distance Functions

## 4.1. Introduction

In this chapter, we prove a theorem that extends Looijenga's transversality theorem, which applies to a single distance function associated to an embedding of a submanifold, to multiple distance functions associated to such an embedding. Unlike Looijenga's result, our transversality theorem will require that multiple transversality conditions for different distance functions be satisfied at the same points on a hypersurface.

In Section 4.2.1, we define families of *multi-distance functions* to which the transversality theorem applies, and introduce a special kind of multijet extension of these functions in Section 4.2.2. Section 4.3 contains the statement of the theorem as well as an outline of how to prove it. The proof of the theorem, which constitutes the remainder of the chapter, uses the method for a jet mapping into a subspace of multijet space and will rely on appropriate "relative" and "absolute" transversality theorems. It will involve constructing a parametrized family of perturbations of an initial embedding and showing that a multijet extension of the family is transverse to certain submanifolds of the subspace of jet space, ensuring that a residual set of members of the family will be transverse to submanifolds in the subspace.

In Chapter 5, we shall use the transversality theorem to establish the generic linking classification via transversality conditions.

## 4.2. Families of multi-distance functions

**4.2.1. Definition of a multi-distance function.** In this section, we shall introduce the notion of a *multi-distance function* as a means of capturing distance from more than one distinct point in the ambient space to the same point on a boundary hypersurface.

The primary motivation for introducing such a function is to examine the situation when a hypersurface point simultaneously corresponds to a point on the internal medial axis of the region the hypersurface bounds, as well as to a point on the linking medial axis of the complementary region.

Let $X = \coprod_{i=1}^{q} X_i$, where each $X_i$ is a smooth, $n$-dimensional, compact, connected, orientable manifold, and let $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$. The map $\phi$ restricts to the $q$ embeddings $\phi|_{X_i} = \phi_i : X_i \hookrightarrow \mathbb{R}^{n+1}, i = 1, \ldots, q$, with each $\phi_i(X_i) = \mathcal{B}_i$ a compact connected hypersurface bounding a region $\Omega_i$ and satisfying $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$.

Let $X^{(r)} = \underbrace{X \times \ldots \times X}_{r \text{ times}} \setminus \Delta X$, where $\Delta X$ is the generalized diagonal in $X^r$, i.e.,

$$\Delta X = \{(x_1, \ldots, x_r) \in X^r : x_i = x_j \text{ for some } i \neq j\}.$$

Similarly, let $(\mathbb{R}^{n+1})^{(q+1)} = \underbrace{\mathbb{R}^{n+1} \times \ldots \times \mathbb{R}^{n+1}}_{q+1 \text{ times}} \setminus \Delta\mathbb{R}^{n+1}$. We order the $q+1$ distinct copies of $\mathbb{R}^{n+1}$ as $\mathbb{R}_0^{n+1} \times \ldots \times \mathbb{R}_q^{n+1}$. For each $i$, let

$$\pi_i : X \times (\mathbb{R}^{n+1})^{(q+1)} \to X \times \mathbb{R}_i^{n+1}$$

be the natural projection, and let

$$\begin{aligned} \sigma : X \times \mathbb{R}^{n+1} &\to \mathbb{R}, \\ (x, u) &\mapsto ||\phi(x) - u||^2 \end{aligned}$$

be the distance squared function.

**Definition 4.2.1.** *For every $i = 1, \ldots, q$, the **family of multi-distance functions associated to the embedding** $\phi_i$, denoted $\rho_{\phi_i}$, is given by*

$$\begin{aligned} \rho_{\phi_i} : X_i \times (\mathbb{R}^{n+1})^{(q+1)} &\to \mathbb{R}^2, \\ (x_i, u_0, \ldots, u_q) &\mapsto (\sigma \circ \pi_0,\, \sigma \circ \pi_i)(x_i, u_0, \ldots, u_q) \\ &= (||\phi_i(x_i) - u_0||^2,\, ||\phi_i(x_i) - u_i||^2). \end{aligned} \tag{4.1}$$

*In general, the* ***family of multi-distance functions associated to the embedding*** *$\phi$, denoted $\rho_\phi$, is the family*

$$(4.2) \qquad \rho_\phi : X \times (\mathbb{R}^{n+1})^{(q+1)} \to \mathbb{R}^2,$$

$$(x_i, u_0, \ldots, u_q) \mapsto (||\phi_i(x_i) - u_0||^2, \, ||\phi_i(x_i) - u_i||^2),$$

*where $x_i \in X_i$. Observe that $X \times (\mathbb{R}^{n+1})^{(q+1)} = \coprod_{i=1}^{q} \big(X_i \times (\mathbb{R}^{n+1})^{(q+1)}\big)$, so that the restriction $\rho_\phi | (X_i \times \big(\mathbb{R}^{n+1})^{(q+1)}\big) = \rho_{\phi_i}$.*

Thus, the functions are "multi-distance" functions in the sense that they incorporate multiple functions capturing the squared distance from distinct points in the ambient space. The transversality theorem will apply to these families of multi-distance functions. Since the transversality theorem is a result on the level of jet bundles, our focus in the next section is on taking a special kind of multijet of such functions.

**4.2.2. Partial multijet spaces and multi-distance functions.** In Section 1.3 in Chapter 1, we introduced the notions of multijet space and the usual multijet extension of a mapping. In this section, we shall introduce the notion of a *partial multijet extension* for the family $\rho_\phi$ as a means of specifying the number of points at which the multijet is taken on each of the $q$ hypersurfaces that comprise $X$.

For any $s \in \mathbb{Z}^+$, the usual $s$-multi $k$-jet extension of $\rho_\phi$ is

$$(4.3) \qquad {}_sj_1^k(\rho_\phi) : X^{(s)} \times (\mathbb{R}^{n+1})^{(q+1)} \to {}_sJ^k(X, \mathbb{R}^2),$$

$$(x_1, \ldots, x_s, u_0, \ldots, u_q) \to {}_sj_1^k\rho_\phi(\cdot, u_0, \ldots, u_q)(x_1, \ldots, x_s),$$

where the subscript "1" in ${}_sj_1^k(\rho_\phi)$ indicates that the jet is taken with respect to the coordinates on $X$. Consider an ordered partition of the form $s = \ell_1 + \ldots + \ell_q$ with each integer $\ell_i \geq 0$, and let $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$. Let

$$(4.4) \qquad X^{(\boldsymbol{\ell})} = X_1^{(\ell_1)} \times \ldots \times X_q^{(\ell_q)}$$

denote a subset of $X^{(s)}$, where we delete any terms in the product for which $\ell_i = 0$. Also, if $q(\boldsymbol{\ell})$ denotes the number of nonzero $\ell_i$ in the partition of $s$, then let

$$(\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \tag{4.5}$$

denote the subset of $(\mathbb{R}^{n+1})^{(q+1)}$ where we delete any term $\mathbb{R}^{n+1}_i$, $i = 1, \ldots, q$, in the product for which $\ell_i = 0$.

We are interested in restricting the mapping in (4.3) to the space $X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$. For each $i = 1, \ldots, q$, we denote $\boldsymbol{x}_i \in X_i^{(\ell_i)}$ by $\boldsymbol{x}_i = (x_{i_1}, \ldots, x_{i_{\ell_i}})$ for $x_{i_j} \in X_i$ for each $j = 1, \ldots, \ell_i$. When restricting the multijet extension mapping in (4.3) to the space $X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$, we refer to the resulting mapping as the *partial $\boldsymbol{\ell}$-multi $k$-jet* of $\rho_\phi$.

**Definition 4.2.2.** *Let $s = \ell_1 + \ldots + \ell_q$, $\ell_i \geq 0$, be an ordered partition of $s \in \mathbb{Z}^+$, and let $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$. The **partial $\boldsymbol{\ell}$–multi k-jet extension** of $\rho_\phi$ is given by*

$$\begin{aligned} &{}_{\boldsymbol{\ell}}j_1^k \rho_\phi : X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \to {}_sJ^k(X, \mathbb{R}^2) \\ &(\boldsymbol{x}, \boldsymbol{u}) := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_q, u_0, \ldots, u_q) \mapsto ({}_{\ell_1}j_1^k \rho_{\phi_1}(\cdot, u_0, u_1)(\boldsymbol{x}_1), \ldots, {}_{\ell_q}j_1^k \rho_{\phi_q}(\cdot, u_0, u_q)(\boldsymbol{x}_q)). \end{aligned} \tag{4.6}$$

Next, we define a subspace of ${}_sJ^k(X, \mathbb{R}^2)$ called the *partial $\boldsymbol{\ell}$-multi $k$-jet subspace.*

**Definition 4.2.3.** *For $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$ an ordered partition of $s \in \mathbb{Z}^+$, the **partial $\boldsymbol{\ell}$-multi $k$-jet subspace** of ${}_sJ^k(X, \mathbb{R}^2)$ is*

$$ {}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2) := \prod_{i=1}^{q} {}_{\ell_i}J^k(X_i, \mathbb{R}^2). \tag{4.7}$$

**Lemma 4.2.4.**

*(a) ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ is a smooth submanifold of ${}_sJ^k(X, \mathbb{R}^2)$.*

*(b) ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ is a locally trivial fiber bundle over $X^{(\boldsymbol{\ell})}$.*

Proof. For (a), let

$$\pi_1^{(s)} : {}_sJ^k(X, \mathbb{R}^2) \to X^{(s)}$$

denote the multijet source map for ${}_sJ^k(X,\mathbb{R}^2)$. Since $X^{(\boldsymbol{\ell})}$ is an open subset of $X^{(s)}$ and $\pi_1^{(s)}$ is continuous, $(\pi_1^{(s)})^{-1}(X^{(\boldsymbol{\ell})}) = {}_{\boldsymbol{\ell}}E^{(k)}(X,\mathbb{R}^2)$ is an open subset of the smooth manifold ${}_sJ^k(X,\mathbb{R}^2)$. Therefore, ${}_{\boldsymbol{\ell}}E^{(k)}(X,\mathbb{R}^2)$ is a smooth submanifold of ${}_sJ^k(X,\mathbb{R}^2)$.

To prove (b), let $(z_1,\dots,z_q) \in {}_{\boldsymbol{\ell}}E^{(k)}(X,\mathbb{R}^2)$. For $z_i \in {}_{\ell_i}J^k(X_i,\mathbb{R}^2), i = 1,\dots,q$, let $\boldsymbol{x}_i = (x_{i_1},\dots,x_{i_{\ell_i}})$ be its source points. Then, for $i = 1,\dots,q$, we may choose

$$U_i = \prod_{j=1}^{\ell_i} U_{i_j}, \tag{4.8}$$

where $U_{i_j}$ is a coordinate neighborhood of $x_{i_j}$ for every $j = 1,\dots,\ell_i$ and $U_{i_j} \cap U_{i_k} = \emptyset$ for $j \neq k$, so that $U_i \subset X_i^{(\ell_i)}$. Then if $\pi_1^{(\ell_i)} : {}_{\ell_i}J^k(X_i,\mathbb{R}^2) \to X_i^{(\ell_i)}$ denotes the bundle projection map for $i = 1,\dots,q$, we know that

$$(\pi_1^{(\ell_i)})^{-1}(U_i) \cong U_i \times \left(\mathbb{R}^2 \times J^k(n,2)\right)^{\ell_i}$$

as ${}_{\ell_i}J^k(X_i,\mathbb{R}^2)$ is a locally trivial fiber bundle over $X_i^{(\ell_i)}$ with fiber $\left(\mathbb{R}^2 \times J^k(n,2)\right)^{\ell_i}$.

Therefore, let

$$U^{(\boldsymbol{\ell})} = \prod_{i=1}^{q} U_i \ \subset X^{(\boldsymbol{\ell})}, \tag{4.9}$$

so that if

$$\pi_1^{(\boldsymbol{\ell})} : {}_{\boldsymbol{\ell}}E^{(k)}(X,\mathbb{R}^2) \to X^{(\boldsymbol{\ell})}$$

denotes the projection map,

$$(\pi_1^{(\boldsymbol{\ell})})^{-1}(U^{(\boldsymbol{\ell})}) \cong U^{(\boldsymbol{\ell})} \times \prod_{i=1}^{q} \left(\mathbb{R}^2 \times J^k(n,2)\right)^{\ell_i}. \tag{4.10}$$

Thus, ${}_{\boldsymbol{\ell}}E^{(k)}(X,\mathbb{R}^2) \to X^{(\boldsymbol{\ell})}$ is a locally trivial fiber bundle with fiber $\displaystyle\prod_{i=1}^{q} \left(\mathbb{R}^2 \times J^k(n,2)\right)^{\ell_i}$ over the point $\boldsymbol{x} = (\boldsymbol{x}_1,\dots,\boldsymbol{x}_q) \in X^{(\boldsymbol{\ell})}$. $\square$

From (4.6), we observe that the partial multijet ${}_{\boldsymbol{\ell}}j_1^k\rho_\phi(\boldsymbol{x},\boldsymbol{u})$ belongs to ${}_{\boldsymbol{\ell}}E^{(k)}(X,\mathbb{R}^2)$ for all $\phi \in \mathrm{DEmb}(X,\mathbb{R}^{n+1}) \subset \mathrm{Emb}(X,\mathbb{R}^{n+1})$ and for all $(\boldsymbol{x},\boldsymbol{u}) \in X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$. The transversality theorem that we prove in this chapter holds for a residual subset of $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$, and the transversality is relative to the submanifold ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$. Specifically, we require transversality to the special class of closed Whitney stratified subsets $W$ of ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ consisting of strata which are invariant under the action of the group ${}_{\boldsymbol{\ell}}\mathcal{R}^+$.

## 4.3. The transversality theorem

In this section, we state the transversality theorem and provide an outline of its proof. The essential point of the theorem is that the transversality conditions hold for a subspace of mappings obtained by applying an operation, and the transversality is relative to a submanifold of jet space.

### 4.3.1. Statement of the transversality theorem.

**Theorem 4.3.1.** *Let $X = \coprod_{i=1}^{q} X_i$ where each $X_i$ is a smooth, $n$-dimensional, compact, connected, orientable manifold. For $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$, let $W$ be a closed Whitney stratified subset of ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ that is ${}_{\boldsymbol{\ell}}\mathcal{R}^+$-invariant.*

*(a) Let $Z \subset X^{(\boldsymbol{\ell})}$ be compact. Then the set*

$$\mathscr{W} = \{\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1}) : {}_{\boldsymbol{\ell}}j_1^k \rho_\phi \,\overline{\pitchfork}\, W \text{ in } {}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2) \text{ on } Z \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}\}$$

*is an open dense subset for the regular $\mathrm{C}^\infty$ topology.*

*(b) The set*

$$\mathscr{W} = \{\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1}) : {}_{\boldsymbol{\ell}}j_1^k \rho_\phi \,\overline{\pitchfork}\, W \text{ in } {}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2) \text{ on } X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}\}$$

*is a residual subset for the regular $\mathrm{C}^\infty$ topology.*

Note that in Theorem 4.3.1, $(b)$ is a consequence of $(a)$. We may cover $X^{(\boldsymbol{\ell})}$ with countably many compact sets $Z_j$ so that

$$S_j = \{\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1}) : {}_{\boldsymbol{\ell}}j_1^k \rho_\phi \text{ is transverse on } Z_j \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \text{ to } W\}$$

is open and dense in the C$^\infty$ topology by (a); therefore,

$$\mathcal{W} = \{\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1}) : {}_{\boldsymbol{\ell}}j_1^k \rho_\phi \text{ is transverse on } X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \text{ to } W\} = \bigcap_j S_j$$

is residual.

Before beginning the proof, we shall first give an overview of the steps that are involved.

**4.3.2. Method of proof.** The transversality theorem combines elements of Looijenga's transversality theorem with elements of relative and absolute versions of Thom's transversality theorem due to Damon [**12**]. These theorems involve performing an operation on a particular space of mappings and obtaining a transversality statement for the resulting mappings relative to a subspace of jet space.

In general, let $\mathcal{H}$ be a Baire space which is a subspace of $C^\infty$ mappings, and let $\Psi : \mathcal{H} \to C^\infty(M, N)$ be continuous with $M$ and $N$ smooth manifolds. The multijet version of the relative transversality theorem requires $\mathcal{H}$ to have *smooth image in* ${}_sJ^k(M,N)$ in the following sense.

**Definition 4.3.2.** [**12**] *The space of mappings $\mathcal{H}$ is said to have **smooth image in** ${}_sJ^k(M,N)$ provided that the following two conditions are satisfied:*
*(a) ${}_s\mathcal{H}^{(k)} \subset {}_sJ^k(M,N)$ is a smooth submanifold, where*

$${}_s\mathcal{H}^{(k)} = \{{}_sj^k(\Psi(f))(x_1, \ldots, x_s) : f \in \mathcal{H}, (x_1, \ldots, x_s) \in M^{(s)}\};$$

*(b) Given points $\boldsymbol{x} = (x_1, \ldots, x_s) \in M^{(s)}$, $\boldsymbol{y} = (y_1, \ldots, y_s) \in N^s$, there are open sets $\boldsymbol{x} \in U \subset M^{(s)}, \boldsymbol{y} \in V \subset N^s$ such that for every map $h \in \mathcal{H}$, there exists a finite-dimensional smooth manifold $\mathcal{V} \subset \mathcal{H}$ with $h \in \mathcal{V}$ such that the family*

$$\Gamma : M \times \mathcal{V} \to N,$$

$$(x, f) \mapsto \Psi(f)(x)$$

*satisfies the property that the s-multi k-jet extension*

$$ {}_sj^k\Gamma : U \times \mathcal{V} \to {}_sJ^k(U, N), $$

$$ (x_1, \ldots, x_s, f) \mapsto {}_sj^k(\Psi(f))(x_1, \ldots, x_s) $$

*is a smooth submersion onto* ${}_s\mathcal{H}^{(k)} \cap J^k(U, V)$ *at all points* $\{(\boldsymbol{x}, h)\}$ *for* $\boldsymbol{x} \in U \cap (\Psi(h)^s)^{-1}(V)$.

**Theorem 4.3.3** (Relative Multi-Transversality Theorem)**.** [**12**, Corollary 1.9] *Suppose* $X \subset M^{(s)}$ *and* $W \subset {}_s\mathcal{H}^{(k)}$ *are closed Whitney stratified subsets. If* $\mathcal{H}$ *has smooth image in* ${}_sJ^k(M, N)$*, then the set*

$$ {}_s\mathcal{W} = \{f \in \mathcal{H} : {}_sj^k\Psi(f) \text{ is transverse to } W \text{ in } {}_s\mathcal{H}^{(k)} \text{ on } X\} $$

*is residual in the* $\mathrm{C}^\infty$ *topology. If* $X$ *is compact, the set is open and dense.*

For the absolute version of Damon's transversality theorem, let $W \subset {}_sJ^k(M, N)$ be either a Whitney stratified submanifold, or a submanifold whose closure $\overline{W}$ is Whitney stratified with $W$ as one of the strata; in the second case, the submanifold is said to be *relatively Whitney stratifiable.* The next definition explains what it means for $\Psi$ to be either *transverse* to $W$ or *completely transverse* to $W$ (i.e., transverse to $\overline{W}$).

**Definition 4.3.4.** [**12**] *The map* $\Psi$ *is said to be* ***transverse*** *or* ***completely transverse*** *to* $W \subset {}_sJ^k(M, N)$ *if, given an open subset* $\mathcal{U} \subset \mathcal{H}$*,* $\boldsymbol{x} = (x_1, \ldots, x_s) \in M^{(s)}$ *and* $\boldsymbol{y} = (y_1, \ldots, y_s) \in N^s$*, there are open sets* $\boldsymbol{x} \in U \subset M^{(s)}, \boldsymbol{y} \in V \subset N^s$ *such that for every map* $h \in \mathcal{U}$*, there exists a finite-dimensional smooth manifold* $\mathcal{V} \subset \mathcal{U}$ *with* $h \in \mathcal{V}$ *such that the family*

$$ \Gamma : M \times \mathcal{V} \to N, $$

$$ (x, f) \mapsto \Psi(f)(x) $$

*satisfies the property that the s-multi k-jet extension*

$$
{}_sj^k\Gamma : U \times \mathcal{V} \to {}_sJ^k(U, N),
$$
$$
(x_1, \dots, x_s, f) \mapsto {}_sj^k(\Psi(f))(x_1, \dots, x_s)
$$

*is transverse or completely transverse to* $W \cap J^k(U, V)$ *at all points* $\{(\boldsymbol{x}, h)\}$ *with* $\boldsymbol{x} \in U \cap (\Psi(h)^s)^{-1}(V)$.

**Theorem 4.3.5** (Absolute Multi-Transversality Theorem)**.** [**12**, Corollary 1.11] *Suppose* $X \subset M^{(s)}$ *and* $W \subset {}_s\mathcal{H}^{(k)}$ *are closed Whitney stratified subsets, and suppose* $\Psi$ *is transverse to* $W$ *in* ${}_sJ^k(M, N)$*. Then the sets*

$$
{}_s\mathcal{W} = \{f \in \mathcal{H} : {}_sj^k\Psi(f) \textit{ is transverse to } W \textit{ in } {}_sJ^k(M, N) \textit{ on } X\}
$$

*and*

$$
{}_s\mathcal{W}' = \{f \in \mathcal{H} : {}_sj^k\Psi(f) \textit{ is completely transverse to } W \textit{ in } {}_sJ^k(M, N) \textit{ on } X\}
$$

*are residual in the* $\mathrm{C}^\infty$ *topology. If* $X$ *is compact, the set is open and dense.*

To obtain the new transversality results in this chapter, we must apply the following hybrid version of Theorems 4.3.3 and 4.3.5.

**Theorem 4.3.6** (Hybrid Multi-Transversality Theorem)**.** *Let* $X \subset M^{(s)}$ *and* $W \subset {}_s\mathcal{H}^{(k)}$ *be closed Whitney stratified subsets, and suppose the continuous map* ${}_sj^k\Psi : \mathcal{H} \to {}_sJ^k(M, N)$ *maps into the subbundle* ${}_s\mathcal{H}^{(k)}$*. Suppose further that the map* $\Psi$ *is transverse to* $W$ *in the sense of Theorem 4.3.5. Then the set*

$$
{}_s\mathcal{W} = \{f \in \mathcal{H} : {}_sj^k\Psi(f) \textit{ is transverse to } W \textit{ in } {}_s\mathcal{H}^{(k)} \textit{ on } X\}
$$

*is a residual set in the* $\mathrm{C}^\infty$ *topology. If* $X$ *is compact, the set is open and dense.*

Applying Theorem 4.3.6 to obtain our transversality results requires the following steps.

(1) Define

$$\Psi : \mathrm{DEmb}(X, \mathbb{R}^{n+1}) \to C^\infty\left(X \times (\mathbb{R}^{n+1})^{(q+1)}, \mathbb{R}^2\right) \tag{4.11}$$
$$\phi \mapsto \rho_\phi,$$

where $\rho_\phi$ is the multi-distance function in Definition 4.2.1. We first show that the map $\Psi$ is continuous.

(2) We then show that $\Psi$ is transverse to any submanifold $W$ of ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ that is ${}_{\boldsymbol{\ell}}\mathcal{R}^+-$invariant.

To do so, let $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ and let $(\boldsymbol{x}, \boldsymbol{u}) := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_q, u_0, \ldots, u_q) \in U^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$ with $U^{(\boldsymbol{\ell})}$ as in (4.9). In the notation of Definition 4.3.4, our $N$ in this case is $\mathbb{R}^2$, so we simply take $V = (\mathbb{R}^2)^s$ as $N$ is already Euclidean space. We must find a finite-dimensional smooth manifold $\mathcal{V} \subset \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ with $\phi \in \mathcal{V}$ such that the family

$$\Gamma : X \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \times \mathcal{V} \to \mathbb{R}^2$$
$$(x, \boldsymbol{u}, f) \mapsto \Psi(f)(x, \boldsymbol{u}) = \rho_f(x, \boldsymbol{u})$$

induces a partial multi $k$-jet extension

$${}_{\boldsymbol{\ell}}j_1^k\Gamma : X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \times \mathcal{V} \to {}_sJ^k(X, \mathbb{R}^2), \tag{4.12}$$
$$(\boldsymbol{x}, \boldsymbol{u}, f) \mapsto {}_{\boldsymbol{\ell}}j_1^k\rho_f(\boldsymbol{x}, \boldsymbol{u})$$

which, at points $\{(\boldsymbol{x}, \boldsymbol{u})\} \times \{\phi\} \in U^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \times \mathcal{V}$, is transverse to $W$ inside ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$. To find such a finite-dimensional manifold $\mathcal{V}$, we modify the method used for constructing a family of perturbations of an initial embedding for the case of a single distance function.

(3) Then, we compute the derivative of the map ${}_{\boldsymbol{\ell}}j_1^k\Gamma$ by taking the derivatives with respect to the local coordinates on the surfaces $X_i$, the coordinates in $(\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$, and the parameters in $\mathcal{V}$.

(4) The final step is to use these derivatives to verify that the map in (4.12) is transverse at the indicated points to submanifolds of ${}_{\boldsymbol{\ell}}E^{(k)}(U,\mathbb{R}^2)$ with the ${}_{\boldsymbol{\ell}}\mathcal{R}^{+}-$invariance property. Upon establishing the fact that $\Psi$ satisfies the required transversality result, Theorem 4.3.1 will follow from an application of the Hybrid Multi-Transversality Theorem.

## 4.4. Continuity of $\Psi$

Proof. Let $\phi : X = \coprod_{1}^{q} X_i \hookrightarrow \mathbb{R}^{n+1}$ be an embedding which restricts to the collection of embeddings $\phi_i : X_i \hookrightarrow \mathbb{R}^{n+1}, i = 1, \dots, q$. To show that the map $\Psi$ in (4.11) is continuous, we shall require one property of continuous mappings in the Whitney $\mathrm{C}^\infty$ topology, which we recall from Proposition 3.5 in Chapter II of [**23**].

**Lemma 4.4.1.** *Let $X, Y, Z$ be smooth manifolds. If $\phi : Y \to Z$ is a smooth map, the mapping*

$$\phi_* : \mathrm{C}^\infty(X,Y) \to \mathrm{C}^\infty(X,Z),$$
$$f \mapsto \phi \circ f$$

*is continuous in the* $\mathrm{C}^\infty$ *topology.*

Using this, we obtain:

**Proposition 4.4.2.** *The map* $\Psi$ *in* (4.11) *is continuous in the* $\mathrm{C}^\infty$ *topology.*

Proof. It is sufficient to show that, for each $i = 1, \dots, q$, the map

$$\Psi_i : \mathrm{DEmb}(X_i, \mathbb{R}^{n+1}) \to \mathrm{C}^\infty\left(X_i \times (\mathbb{R}^{n+1})^{(q+1)}, \mathbb{R}^2\right), \ \phi_i \mapsto \rho_{\phi_i}$$

is continuous. Let $id : (\mathbb{R}^{n+1})^{(q+1)} \to (\mathbb{R}^{n+1})^{(q+1)}$ denote the identity map. For any $i = 1, \ldots, q$, the mapping

$$\begin{aligned} \mathrm{C}^\infty(X_i, \mathbb{R}^{n+1}) &\to \mathrm{C}^\infty\left(X_i \times (\mathbb{R}^{n+1})^{(q+1)}, \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^{(q+1)}\right), \\ f &\mapsto f \times id \end{aligned}$$

is continuous. Hence, on the open subset $\mathrm{DEmb}(X_i, \mathbb{R}^{n+1})$, the mapping

$$\begin{aligned} \psi_i : \mathrm{DEmb}(X_i, \mathbb{R}^{n+1}) &\to \mathrm{C}^\infty\left(X_i \times (\mathbb{R}^{n+1})^{(q+1)}, \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^{(q+1)}\right), \\ \phi_i &\mapsto \phi_i \times id \end{aligned}$$

is continuous. Define for any $i = 1, \ldots, q$ the mapping

$$\begin{aligned} R_i : \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^{(q+1)} &\to \mathbb{R}^2, \\ (x, u_0, \ldots, u_q) &\mapsto (||x - u_0||^2, \, ||x - u_i||^2), \end{aligned}$$

where $u_j \in \mathbb{R}^{n+1}_j$ for all $j$. Since $R_i$ is smooth, by Lemma 4.4.1, the mapping

$$\begin{aligned} R_{i*} : \mathrm{C}^\infty\left(X_i \times (\mathbb{R}^{n+1})^{(q+1)}, \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^{(q+1)}\right) &\to \mathrm{C}^\infty\left(X_i \times (\mathbb{R}^{n+1})^{(q+1)}, \mathbb{R}^2\right), \\ f &\mapsto R_i \circ f \end{aligned}$$

is continuous. Hence, since $\Psi_i = R_{i*} \circ \psi_i$ for every $i$, $\Psi$ is continuous in the $\mathrm{C}^\infty$ topology. $\square$

The remainder of the proof focuses on establishing transversality of the map $\Psi$ to any ${}_{\ell}\mathcal{R}^+$−invariant submanifold $W$ of the partial multijet space.

## 4.5. Construction of the families of perturbations

Given an embedding $\phi : X \to \mathbb{R}^{n+1}$, we construct a finite family of perturbations needed for (4.12). We motivate this by first explaining how to construct a family of perturbations of an initial embedding to establish Looijenga's transversality theorem.

The method we employ to find our family of perturbations $\mathcal{V}$ extends the method used in the case of a single distance function to the multi-distance setting.

**Perturbation family for a single distance function.** Let $V$ denote the vector space of polynomial mappings $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of degree $\leq k$, which have the form

$$(x^{(1)}, \dots, x^{(n+1)}) \mapsto (g_1(x^{(1)}, \dots, x^{(n+1)}), \dots, g_{n+1}(x^{(1)}, \dots, x^{(n+1)})),$$

where, for each $j = 1, \dots, n+1$, the map $g_j : \mathbb{R}^{n+1} \to \mathbb{R}$ is a polynomial of degree $\leq k$. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{n+1})$ and $|\boldsymbol{\alpha}| := \sum_{j=1}^{n+1} \alpha_j$ with $0 \leq |\boldsymbol{\alpha}| \leq k$; define

$$w_{\boldsymbol{\alpha},l} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \tag{4.13}$$

$$\boldsymbol{x} := (x^{(1)}, \dots, x^{(n+1)}) \mapsto (0, \dots, \underbrace{\prod_{j=1}^{n+1} (x^{(j)})^{\alpha_j}}_{l}, 0, \dots, 0),$$

i.e., the monomial $\prod_{j=1}^{n+1} (x^{(j)})^{\alpha_j}$ is in the $l^{th}$ component for $l = 1, \dots, n+1$. The collection $\{w_{\boldsymbol{\alpha},l}\}$, ranging over all possible $\boldsymbol{\alpha}$ with $0 \leq |\boldsymbol{\alpha}| \leq k$ and $l = 1, \dots, n+1$, is a monomial basis for the vector space $V$, which has dimension $d = \binom{n+1+k}{k}(n+1)$. Thus, any $\boldsymbol{v} \in V$ may be written as $\boldsymbol{v} = \sum_{l=1}^{n+1} \sum_{|\boldsymbol{\alpha}| \leq k} v_{\boldsymbol{\alpha},l}\, w_{\boldsymbol{\alpha},l}$, where the $v_{\boldsymbol{\alpha},l}$ are the coefficients of the polynomial mapping.

For each $i \in \{1, \dots, q\}$ and for choices of nonnegative integers $\ell_i$, we shall construct a perturbation of the embedding $\phi_i : X_i \to \mathbb{R}^{n+1}$. We shall use the polynomials defined globally on $\mathbb{R}^{n+1}$ to induce local polynomial deformations of these initial embeddings. We begin with a preliminary lemma.

**Lemma 4.5.1.** *If $V_i$ denotes a sufficiently small open neighborhood of 0 in $V$,*

$$\widetilde{\Phi}_i : X_i \times V_i \to \mathbb{R}^{n+1}, \tag{4.14}$$

$$(x, \boldsymbol{v}) \mapsto \phi_i(x) + (\boldsymbol{v} \circ \phi_i)(x)$$

$$= \phi_i(x) + \sum_{l=1}^{n+1} \sum_{\alpha} v_{\alpha,l} \, (w_{\alpha,l} \circ \phi_i)(x)$$

*is a family of embeddings.*

Proof. Since $\widetilde{\Phi}_i(x, 0) = \phi_i(x)$ and $\phi_i$ is an embedding, and since $\mathrm{Emb}(X, \mathbb{R}^{n+1}) \subset \mathrm{C}^\infty(X, \mathbb{R}^{n+1})$ is an open subset [**23**], there exists a sufficiently small neighborhood $V_i$ of 0 in $V$ so that $\widetilde{\Phi}_i(\cdot, v) : X_i \to \mathbb{R}^{n+1}$ is also an embedding for all $v \in V_i$. Note that the map

$$V \to \mathrm{C}^\infty(X_i, \mathbb{R}^{n+1})$$

$$\boldsymbol{v} \mapsto \widetilde{\Phi}_i(\cdot, \boldsymbol{v})$$

is continuous for the $\mathrm{C}^\infty$ topology on $\mathrm{C}^\infty(X_i, \mathbb{R}^{n+1})$. This amounts to showing that the map $\boldsymbol{v} \mapsto \boldsymbol{v} \circ \phi_i$ is continuous. This follows from Proposition 3.9 in [**23**], which states that if $X, Y, Z$ are smooth manifolds and $X$ is compact, the mapping

$$\mathrm{C}^\infty(X, Y) \times \mathrm{C}^\infty(Y, Z) \to \mathrm{C}^\infty(X, Z),$$

$$f \times g \mapsto g \circ f$$

is continuous in the $\mathrm{C}^\infty$ topology. Here, composition on the right with $\phi_i \in \mathrm{C}^\infty(X_i, \mathbb{R}^{n+1})$ is continuous since $X_i$ is compact. $\square$

Next, for $j = 1, \ldots, \ell_i$, let $V_{i_j}$ denote a copy of the vector space $V_i$ so that

$$\widetilde{\Phi}_{i_j} : X_i \times V_{i_j} \to \mathbb{R}^{n+1} \tag{4.15}$$

defined as in (4.14) is, restricted to a sufficiently small neighborhood of 0, a family of embeddings with $\boldsymbol{v}_{i_j} \in V_{i_j}$. Let

$$K_i = K_{i_1} \times \ldots \times K_{i_{\ell_i}} \tag{4.16}$$

be a compact subset of $X_i^{(\ell_i)}$ satisfying $K_{i_j} \cap K_{i_l} = \emptyset, j \neq l$ with $K_{i_j}$ a compact neighborhood of $x_{i_j}$ for all $j$. Also, let $\widetilde{K}_{i_j}, j = 1, \ldots, \ell_i$, be disjoint compact subsets of $X_i$ such that $K_{i_j} \subset \operatorname{int}(\widetilde{K}_{i_j})$, the interior of $\widetilde{K}_{i_j}$. Then

$$\mathcal{O} = \left\{ X_i \setminus \bigcup_{j=1}^{\ell_i} K_{i_j}, \{\operatorname{int}(\widetilde{K}_{i_j})\}_{j=1}^{\ell_i} \right\}$$

is an open cover of $X_i$. Choose a smooth partition of unity $\lambda_i = \{\lambda_{i_j}\}_{j=1}^{\ell_i}$ subordinate to $\mathcal{O}$ such that $\lambda_{i_j}|K_{i_j} \equiv 1$. Note that we choose the partition of unity in this way since we shall work entirely within the set $\{\operatorname{int}(\widetilde{K}_{i_j})\}_j$.

**Definition 4.5.2.** *Let $\mathcal{V}_i := V_{i_1} \times \ldots \times V_{i_{\ell_i}}$, and define the family of perturbations*

$$\begin{aligned} \Phi_i &: X_i \times \mathcal{V}_i \to \mathbb{R}^{n+1}, \\ (x, \boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_{\ell_i}}) &\mapsto \phi_i(x) + \sum_{j=1}^{\ell_i} \lambda_{i_j}(x)\, (\boldsymbol{v}_{i_j} \circ \phi_i)(x). \end{aligned} \tag{4.17}$$

**Remark 4.5.3.** Observe that $\Phi_i(x, \mathbf{0}) = \phi_i(x)$, and that for $x \in K_{i_j}$, the map reduces to $\widetilde{\Phi}_{i_j}(x, \boldsymbol{v}_{i_j})$ since $\lambda_{i_j}|_{K_{i_j}} \equiv 1$. The fact that $\Phi_i$ is a family of embeddings follows from the reasoning in the proof of Lemma 4.5.1.

**Definition 4.5.4.** *With $\Phi_i$ as in* (4.17)*, the* ***perturbation of the distance function*** *is*

$$\begin{aligned} \sigma_{\Phi_i} &: X_i \times \mathbb{R}^{n+1} \times \mathcal{V}_i \to \mathbb{R}, \\ (x, u, \boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_{\ell_i}}) &\mapsto ||\Phi_i(x, \boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_{\ell_i}}) - u||^2. \end{aligned} \tag{4.18}$$

**Perturbation family for multi-distance functions.** Next, we extend the method described above for a single distance function to multi-distance functions. We now introduce the finite dimensional manifold $\mathcal{V}$ that is required in (4.12) to show that $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ has smooth image in ${}_sJ^k(X, \mathbb{R}^2)$.

**Definition 4.5.5.** *With $\mathcal{V}_i$ as in Definition 4.5.2 for $i = 1, \ldots, q$, define*

$$\mathcal{V} = \mathcal{V}_1 \times \ldots \times \mathcal{V}_q. \tag{4.19}$$

Using this parameter space, we will construct a family of perturbations of the multi-distance functions and compute the induced partial multijet extension of the resulting family.

**Definition 4.5.6.**

*(a) The **perturbation family of the multi-distance functions** is given by*

$$\begin{aligned} &\Gamma : X \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \times \mathcal{V} \to \mathbb{R}^2, \\ &(x_i, u_0, \ldots, u_q, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_q) \mapsto (||\Phi_i(x_i, \boldsymbol{v}_i) - u_0||^2, ||\Phi_i(x_i, \boldsymbol{v}_i) - u_i||^2), \end{aligned} \tag{4.20}$$

*where $x_i \in X_i, \boldsymbol{v}_i \in \mathcal{V}_i$ for $i = 1, \ldots, q$, and $u_j \in \mathbb{R}_j^{n+1}$, $j = 0, \ldots, q$.*

*(b) Let $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_q, u_0, \ldots, u_q, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_q)$, where $\boldsymbol{x}_i \in X_i^{(\ell_i)}$. The **partial $\ell$-multi $k$-jet extension** of $\Gamma$ is*

(4.21)

$$\begin{aligned} &{}_{\boldsymbol{\ell}}j_1^k(\Gamma) : X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \times \mathcal{V} \to {}_sJ^k(X, \mathbb{R}^2), \\ &(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) \mapsto ({}_{\ell_1}j_1^k(||\Phi_1(\,\cdot\,, \boldsymbol{v}_1) - u_0||^2, ||\Phi_1(\,\cdot\,, \boldsymbol{v}_1) - u_1||^2)(\boldsymbol{x}_1), \ldots, \\ &\qquad {}_{\ell_q}j_1^k(||\Phi_q(\,\cdot\,, \boldsymbol{v}_q) - u_0||^2, ||\Phi_q(\,\cdot\,, \boldsymbol{v}_q) - u_q||^2)(\boldsymbol{x}_q)). \end{aligned}$$

## 4.6. Computation of derivatives

In this section, we compute all partial derivatives of the families of perturbations introduced in the previous section, which are necessary to establish the transversality theorem.

**Computing derivatives in the single distance function case.** From (4.18), we obtain the $\ell_i$-multijet mapping

$$
{}_{\ell_i}j_1^k(\sigma_{\Phi_i}) : X_i^{(\ell_i)} \times \mathbb{R}^{n+1} \times \mathcal{V}_i \to {}_{\ell_i}J^k(X_i, \mathbb{R}), \tag{4.22}
$$

$$
(\boldsymbol{x}_i, u, \boldsymbol{v}_i) := (x_1, \dots, x_{\ell_i}, u, \boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_{\ell_i}}) \mapsto {}_{\ell_i}j_1^k(||\Phi_i(\,\cdot\,, \boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_{\ell_i}}) - u||^2)(x_1, \dots, x_{\ell_i}).
$$

In order to compute the derivative of ${}_{\ell_i}j_1^k(\sigma_{\Phi_i})$ at a point $(\boldsymbol{x}, u, \mathbf{0}) \in K_i \times \mathbb{R}^{n+1} \times \mathcal{V}_i$ with $K_i$ as in (4.16), we introduce a suitable coordinate system. We shall apply a translation and rotation so that $\phi_i(x) \in \mathcal{B}_i = \phi_i(X_i)$ is the origin and $\mathcal{B}_i$ can be locally given in Monge form. That is, if $(x^{(1)}, \dots, x^{(n)})$ define local coordinates on $X_i$ and $\kappa^1, \dots, \kappa^n$ denote the principal curvatures of $\mathcal{B}_i$ at the origin, we may write $\mathcal{B}_i$ as a hypersurface

$$
x^{(n+1)} = f\left(x^{(1)}, \dots, x^{(n)}\right) = \frac{1}{2} \sum_{j=1}^{n} \kappa^j x^{(j)2} + \dots \tag{4.23}
$$

passing through the origin with tangent plane at the origin equal to the plane $x^{(n+1)} = 0$. Then, we may use $x^{(1)}, \dots, x^{(n)}$ as local coordinates for $X_i$ so that

$$
\phi_i = (\phi_i^{(1)}, \dots, \phi_i^{(n+1)}) = (x^{(1)}, \dots, x^{(n)}, f(x^{(1)}, \dots, x^{(n)})).
$$

The distance squared function $\sigma(\cdot, u) : X_i \to \mathbb{R}$ for $u = (u^{(1)}, \dots, u^{(n+1)}) \in \mathbb{R}^{n+1}$ is

$$
\sigma(x, u) = ||\phi_i(x) - u||^2 = \sum_{j=1}^{n+1} (x^{(j)} - u^{(j)})^2, \tag{4.24}
$$

where the last term is $(f(x^{(1)}, \ldots, x^{(n)}) - u^{(n+1)})^2$. Furthermore, each term $(\boldsymbol{v}_{i_j} \circ \phi_i)(x)$ in (4.17) may be written as

$$(\boldsymbol{v}_{i_j} \circ \phi_i)(x) = \sum_{l=1}^{n+1} \sum_{|\boldsymbol{\alpha}| \leq k} v_{j_{\boldsymbol{\alpha}},l} \, (w_{j_{\boldsymbol{\alpha}},l} \circ (x^{(1)}, \ldots, x^{(n+1)})), \tag{4.25}$$

where each monomial $w_{j_{\boldsymbol{\alpha}},l} \circ \phi_i$ is a monomial in $x^{(1)}, \ldots, x^{(n)}$ and $x^{(n+1)} = f(x^{(1)}, \ldots, x^{(n)})$.

Using this coordinate system, we compute the derivative of the map in (4.22), which is explicitly given by:

(4.26)

$$_{\ell_i}j_1^k(||\Phi_i(\,\cdot\,, \boldsymbol{v}) - u||^2)(\boldsymbol{x}_i) = \left(j_1^k(||\Phi_i(\,\cdot\,, \boldsymbol{v}) - u||^2)(x_1), \ldots, j_1^k(||\Phi_i(\,\cdot\,, \boldsymbol{v}) - u||^2)(x_{\ell_1})\right).$$

To do so, we first compute in Proposition 4.6.1 the partial derivatives of each coordinate of the map

$$j_1^k||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x)$$

evaluated at $x = x_j$ and $\boldsymbol{v} = 0$ for any $j \in \{1, \ldots, \ell_1\}$. Then, the derivative of the map in (4.22) follows from the results of Proposition 4.6.1.

In these computations, we view the $k$-jet as a $k^{\text{th}}$ degree polynomial in $\boldsymbol{x}$.

**Proposition 4.6.1.** *With the preceding notation, the partial derivatives of*

$$j_1^k||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x)$$

*evaluated at $\boldsymbol{v} = 0$ and at $x = x_j$ are as follows:*

*(a) For $j = 1, \ldots, n$,*

$$\frac{\partial}{\partial x^{(j)}} \left(j_1^k||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x)\right)\Bigg|_{x=x_j, \boldsymbol{v}=0} = \left(2(x_j^{(j)} - u^{(j)}) + 2j_1^k\left((f - u^{(n+1)}) \cdot \frac{\partial f}{\partial x^{(j)}}\right)\right)\Bigg|_{x=x_j}.$$

*(b) For $j = 1, \ldots, n+1$,*

$$\frac{\partial}{\partial u^{(j)}} \left(j_1^k||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x)\right)\Bigg|_{x=x_j, \boldsymbol{v}=0} = -2\left(j_1^k \phi_i^{(j)}(x) - u^{(j)}\right)\Bigg|_{x=x_j}.$$

*(c) For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n+1})$ with $|\boldsymbol{\alpha}| \le k$ and $l = 1, \ldots, n+1$,*

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}} \left( j_1^k ||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x) \right) \Big|_{x=x_j, \boldsymbol{v}=0} = 2 \left( j_1^{k-|\boldsymbol{\alpha}|} \phi_i^{(l)}(x) - u^{(l)} \right) \left( \prod_{m=1}^{n+1} (x^{(m)})^{\alpha_m} \right) \Big|_{x=x_j}.$$

Proof. To prove (a), since $||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x)$ is smooth, we may interchange the order in which we take the partial derivative with respect to $x^{(j)}$ and the partial derivatives constituting the jet:

$$\frac{\partial}{\partial x^{(j)}} \left( j_1^k ||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x) \right) \Big|_{x=x_j, \boldsymbol{v}=0} = j_1^k \left( \frac{\partial}{\partial x^{(j)}} ||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x) \right) \Big|_{x=x_j, \boldsymbol{v}=0}.$$

Then

$$\begin{aligned}
j_1^k \left( \frac{\partial}{\partial x^{(j)}} ||\Phi_i(\cdot, \boldsymbol{v}) - u||^2(x) \right) \Big|_{\substack{x=x_j \\ \boldsymbol{v}=0}} &= j_1^k \left( \frac{\partial}{\partial x^{(j)}} ||\Phi_i(x, 0) - u||^2 \right) \Big|_{x=x_j} \\
&= j_1^k \left( \frac{\partial}{\partial x^{(j)}} ||\phi_i(x) - u||^2 \right) \Big|_{x=x_j} \\
&= 2 j_1^k \left\langle \phi_i(x) - u, \frac{\partial \phi_i(x)}{\partial x^{(j)}} \right\rangle \Big|_{x=x_j} \\
&= 2 j_1^k \left( \sum_{m=1}^{n+1} \left( \phi_i^{(m)}(x) - u^{(m)} \right) \left( \frac{\partial \phi_i^{(m)}(x)}{\partial x^{(j)}} \right) \right) \Big|_{x=x_j} \\
&= 2 \sum_{m=1}^{n+1} j_1^k \left( \left( \phi_i^{(m)}(x) - u^{(m)} \right) \left( \frac{\partial \phi_i^{(m)}(x)}{\partial x^{(j)}} \right) \right) \Big|_{x=x_j} \\
&= \left( 2(x_j^{(j)} - u^{(j)}) + 2 j_1^k \left( (f - u^{(n+1)}) \cdot \frac{\partial f}{\partial x^{(j)}} \right) \right) \Big|_{x=x_j}.
\end{aligned}$$

The last two lines follow from linearity of $j_1^k$ and from the fact that $\dfrac{\partial \phi_i^{(m)}(x)}{\partial x^{(j)}} = 1$ for $m = j$ and 0 otherwise.

The proof of (b) uses the same reasoning as the above, i.e.,

$$\begin{aligned}\frac{\partial}{\partial u^{(j)}}\left(j_1^k\|\Phi_i(\cdot,\boldsymbol{v})-u\|^2(x)\right)\Big|_{x=x_j,\boldsymbol{v}=0} &= j_1^k\left(\frac{\partial}{\partial u^{(j)}}\|\phi_i(x)-u\|^2\right)\Big|_{x=x_j}\\ &= 2j_1^k\left\langle\phi_i(x)-u,-\frac{\partial u}{\partial u^{(j)}}\right\rangle\Big|_{x=x_j}\\ &= -2j_1^k(\phi_i^{(j)}(x)-u^{(j)})\Big|_{x=x_j}.\end{aligned}$$

To prove (c), we first observe that the map

$$\Phi_i(x,\boldsymbol{v})=\phi_i(x)+\sum_{j=1}^{\ell_i}\lambda_{i_j}(x)\,(\boldsymbol{v}_{i_j}\circ\phi_i)(x)$$

reduces to $\widetilde{\Phi}_i(x,\boldsymbol{v}_{i_j})$ for $x_j\in\operatorname{int}(K_{i_j})$ since $\lambda_{i_j}|K_{i_j}\equiv 1$. Therefore, using the notation of (4.25),

$$\begin{aligned}\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\Phi_i(x,\boldsymbol{v}))\Big|_{\boldsymbol{v}=0} &= \frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}\left(\phi_i(x)+\sum_{l=1}^{n+1}\sum_{|\boldsymbol{\alpha}|\le k}v_{j_{\boldsymbol{\alpha}},l}(w_{j_{\boldsymbol{\alpha}},l}\circ\phi_i)(x)\right)\Big|_{\boldsymbol{v}=0}\\ &= (w_{j_{\boldsymbol{\alpha}},l}\circ\phi_i)(x).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}\left(j_1^k\|\Phi_i(\cdot,\boldsymbol{v})-u\|^2(x)\right)\Big|_{x=x_j,\boldsymbol{v}=0} &= j_1^k\left(\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}\left\langle\Phi_i(\cdot,\boldsymbol{v})-u,\Phi_i(\cdot,\boldsymbol{v})-u\right\rangle\right)(x)\Big|_{\substack{\boldsymbol{x}=\boldsymbol{x_j}\\ \boldsymbol{v}=0}}\\ &= 2j_1^k\left\langle\phi_i(x)-u,(w_{j_{\alpha},l}\circ\phi_i)\right\rangle\Big|_{x=x_j}\\ &= 2j_1^k\left((\phi_i^{(l)}(x)\Big|_{x=x_j}-u^{(l)})(\prod_{m=1}^{n+1}(x^{(m)})^{\alpha_m}\right)\\ &= 2\left((j_1^{k-|\alpha|}\phi_i^{(l)}(x)\Big|_{x=x_j}-u^{(l)})(\prod_{m=1}^{n+1}(x^{(m)})^{\alpha_m}\right)\end{aligned}$$

by linearity and the fact that the monomial $\prod_{m=1}^{n+1}(x^{(m)})^{\alpha_m}$ has degree $|\alpha|$. Specifically, if $|\alpha|<k$ and $l\neq n+1$, $j_1^{k-|\alpha|}(\phi_i^{(l)})=x^{(l)}$ since the $(k-|\alpha|)$-jet of a polynomial of degree

$\leq k - |\alpha|$ is itself. If $|\alpha| = k$ and $l \neq n+1$, $j_1^{k-|\alpha|}(\phi_i^{(l)}) = 0$, and when $l = n+1$, $j_1^{k-|\alpha|}(\phi_i^{(l)}) = j_1^{k-|\alpha|}(f(x^{(1)}, \ldots, x^{(n)}))$. $\square$

**Computing derivatives in the multi-distance function case.** In this section, we once again use Monge coordinates to compute the derivative of the partial multijet extension mapping in (4.21). We always choose local coordinates $(x^{(1)}, \ldots, x^{(n)})$ separately about each point $x_{i_j} \in X_i$ so that in each case, $\phi_i(x_{i_j})$ is the origin and the tangent plane at the origin is the coordinate hyperplane $x^{(n+1)} = 0$.

Let $K_i = K_{i_1} \times \ldots \times K_{i_{\ell_i}}$ be defined as in (4.16), and let

$$K = K_1 \times \ldots \times K_q \tag{4.27}$$

so that $K$ is a compact subset of $X^{(\boldsymbol{\ell})}$. We shall assume that $U_i = \prod_{j=1}^{\ell_i} U_{i_j} \subset K_i$ so that $U_{i_j}$ is an open subset of $K_{i_j}$; thus, $U^{(\boldsymbol{\ell})} = \prod_{i=1}^{q} U_i \ \subset K$. We shall compute the derivative of

$$\begin{aligned} \beta_i : X_i \times (\mathbb{R}_0^{n+1} \times \mathbb{R}_i^{n+1}) \setminus \Delta \times \mathcal{V}_i &\to J^k(X_i, \mathbb{R}^2), \\ (x,\, u_0,\, u_i,\, \boldsymbol{v}_i) &\mapsto j_1^k \left(||\Phi_i(\cdot, \boldsymbol{v}_i) - u_0||^2, ||\Phi_i(\cdot, \boldsymbol{v}_i) - u_i||^2\right)(x) \end{aligned} \tag{4.28}$$

at any point $(x_{i_j}, u_0, u_i, \mathbf{0}) \in U_{i_j} \times (\mathbb{R}_0^{n+1} \times \mathbb{R}_i^{n+1}) \setminus \Delta\ \times \mathcal{V}_i$ for any $j = 1, \ldots, \ell_i$. (As in Remark 4.5.3, since $x_{i_j} \in K_{i_j}$ by assumption, $\beta_i$ actually maps to

$$j_1^k \left(||\widetilde{\Phi}_i(\cdot, \boldsymbol{v}_{i_j}) - u_0||^2, ||\widetilde{\Phi}_i(\cdot, \boldsymbol{v}_{i_j}) - u_i||^2\right)(x)$$

as defined in (4.14), with $\boldsymbol{v}_{i_j} \in V_{i_j}$.) We may then determine the derivative of

$${}_{\ell_i}j_1^k \left(||\Phi_i(\cdot, \boldsymbol{v}_i) - u_0||^2, ||\Phi_i(\cdot, \boldsymbol{v}_i) - u_i||^2\right)(\boldsymbol{x}_i),$$

which, in turn, allows us to determine the derivative of the map in (4.21).

Choosing independent local coordinates about each $x_{i_j}$, the following proposition follows immediately from Proposition 4.6.1.

**Proposition 4.6.2.** *The partial derivatives of $\beta_i$ evaluated at $\boldsymbol{v} = 0$ and $x = x_{i_j}$ are as follows:*

*(a) For $l = 1, \ldots, n$,*

$$\frac{\partial}{\partial x^{(l)}}(\beta_i)\bigg|_{x=x_{i_j},\boldsymbol{v}=0} = \left(\left(2(x^{(l)} - u_0^{(l)}) + 2j_1^k\left((f - u_0^{(n+1)}) \cdot \left(\frac{\partial f}{\partial x^{(l)}}\right)\right)\right)\bigg|_{x=x_{i_j}},\right.$$
$$\left.\left(2(x^{(l)} - u_i^{(l)}) + 2j_1^k\left((f - u_i^{(n+1)}) \cdot \left(\frac{\partial f}{\partial x^{(l)}}\right)\right)\right)\bigg|_{x=x_{i_j}}\right), \tag{4.29}$$

*where $u_0^{(l)}, u_i^{(l)}$ denote the $l^{th}$ coordinates of $u_0$ and $u_i$.*

*(b) For $l = 1, \ldots, n+1$,*

$$\frac{\partial}{\partial u_0^{(l)}}(\beta_i)\bigg|_{x=x_{i_j},\boldsymbol{v}=0} = \left(-2\left(j_1^k\phi_i^{(l)}(x) - u_0^{(l)}\right)\bigg|_{x=x_{i_j}},\ 0\right), \tag{4.30}$$
$$\frac{\partial}{\partial u_i^{(l)}}(\beta_i)\bigg|_{x=x_{i_j},\boldsymbol{v}=0} = \left(0,\ -2\left(j_1^k\phi_i^{(l)}(x) - u_i^{(l)}\right)\bigg|_{x=x_{i_j}}\right).$$

*(c) For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n+1})$ with $|\boldsymbol{\alpha}| \leq k$ and $l = 1, \ldots, n+1$,*

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\beta_i)\bigg|_{x=x_{i_j},\boldsymbol{v}=0} = \left(2\left(j_1^{k-|\boldsymbol{\alpha}|}\phi_i^{(l)}(x) - u_0^{(l)}\right)\left(\prod_{m=1}^{n+1}(x^{(m)})^{\alpha_m}\right)\bigg|_{x=x_{i_j}},\right.$$
$$\left.2\left(j_1^{k-|\boldsymbol{\alpha}|}\phi_i^{(l)}(x) - u_i^{(l)}\right)\left(\prod_{m=1}^{n+1}(x^{(m)})^{\alpha_m}\right)\bigg|_{x=x_{i_j}}\right). \tag{4.31}$$

## 4.7. Completing the proof of Theorem 4.3.1

In this section, we shall begin by proving the theorem that Looijenga needed for his genericity result in the single distance function case. We shall modify the techniques used in the proof of this theorem to prove a corresponding transversality result for multi-distance functions, the final step in the proof of Theorem 4.3.1.

### 4.7.1. Transversality in the single distance function case.

**Theorem 4.7.1.** *With $K_i$ as in (4.16), the mapping*

$$ {}_{\ell_i}j_1^k(\sigma_{\Phi_i}) : X_i^{(\ell_i)} \times \mathbb{R}^{n+1} \times \mathcal{V}_i \to {}_{\ell_i}J^k(X_i, \mathbb{R}), \qquad (\boldsymbol{x}_i, u, \boldsymbol{v}_i) \mapsto {}_{\ell_i}j_1^k(||\Phi_i(\,\cdot\,, \boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_{\ell_i}}) - u||^2)(\boldsymbol{x}_i) \tag{4.32} $$

*is transverse at any point $(\boldsymbol{x}_i, u, \mathbf{0}) \in K_i \times \mathbb{R}^{n+1} \times \mathcal{V}_i$ to any submanifold $W \subset {}_{\ell_i}J^k(X_i, \mathbb{R})$ that is invariant under addition of constants.*

Proof. Let $U_i = \prod_{j=1}^{\ell_i} U_{i_j} \subset K_i$ with $U_{i_j} \subset K_{i_j}$ a coordinate neighborhood of $x_{i_j}$ for all $j = 1, \ldots, \ell_i$, and let $\mathcal{V}_i = \prod_{j=1}^{\ell_i} V_{i_j}$, so that the local version of the map in Theorem 4.7.1 is

$$ \lambda_i : U_i \times \mathbb{R}^{n+1} \times \mathcal{V}_i \to U_i \times \mathbb{R}^{\ell_i} \times J^k(n,1)^{\ell_i}, \qquad (\boldsymbol{x}_i, u, \boldsymbol{v}_i) \mapsto \left( j_1^k(||\widetilde{\Phi}_{i_1}(\,\cdot\,, \boldsymbol{v}_{i_1}) - u||^2)(x_1), \ldots, j_1^k(||\widetilde{\Phi}_{i_{\ell_i}}(\,\cdot\,, \boldsymbol{v}_{i_{\ell_i}}) - u||^2)(x_{\ell_i}) \right), \tag{4.33} $$

with $\widetilde{\Phi}_{i_j}$ as in (4.15) and $\boldsymbol{v}_{i_j} \in V_{i_j}$. We shall prove that, at a point $(\boldsymbol{x}_i, u, \mathbf{0}) \in U_i \times \mathbb{R}^{n+1} \times \mathcal{V}_i$, the map $\lambda_i$: i) is a submersion if $u \neq \phi_i(x_{i_j})$ for any $j$, while ii) if $u = \phi_i(x_{i_j})$ for some $j$, it is transverse to all submanifolds invariant under addition of constants. Note that $u = \phi_i(x_{i_j})$ for at most one value of $j$ since $\phi_i$ is injective.

In what follows, we shall use the algebraic representation $\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$ of the tangent space to $\mathbb{R} \times J^k(n,1)$, which was introduced in Section 1.3 in Chapter 1. Although for the derivative calculations we choose separate local coordinates about each $x_{i_j}$ so that $x_{i_j}$ and $\phi_i(x_{i_j})$ are the origins in each Monge coordinate system, the ideals $\mathfrak{m}_{\boldsymbol{x}}$ will not change.

For i), we first note that $d\lambda_i : T_{\boldsymbol{x}_i}U_i \to T_{\boldsymbol{x}_i}U_i$ is the identity. Next, we consider the projection of $\lambda_i$ onto $\mathbb{R}^{\ell_i} \times J^k(n,1)^{\ell_i}$, and restrict to $\{\boldsymbol{x}_i\} \times \{u\} \times \mathcal{V}_i$, which we identify with $\mathcal{V}_i$. As

$$ \lambda_i : \mathcal{V}_i \to \mathbb{R}^{\ell_i} \times J^k(n,1)^{\ell_i} \tag{4.34} $$

is a product mapping, we can further restrict to each factor, which we denote by

$$\lambda_{i_j} : V_{i_j} \to \mathbb{R} \times J^k(n,1) \tag{4.35}$$

for $j = 1, \ldots, \ell_i$. We identify the target with the ring $\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$. Taking the product over all $j$ then yields that (4.34) is a submersion.

Since $u \neq \phi_i(x_{i_j})$, there is some $l \in \{1, \ldots, n+1\}$ such that $u^{(l)} \neq 0$. Thus, by Proposition 4.6.1, the element $j_1^{k-|\boldsymbol{\alpha}|}\phi_i(x^{(l)}) - u^{(l)}$ is a unit in $\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$. From (4.6.1), we know that

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha},l}}}\left(j_1^k||\Phi_i(\cdot,\boldsymbol{v}) - u||^2(x)\right)\bigg|_{\boldsymbol{v}=0} = 2\left(\left(j_1^{k-|\boldsymbol{\alpha}|}\phi_i(x^{(l)}) - u^{(l)}\right)\left(\prod_{m=1}^{n}(x^{(m)})^{\alpha_m}\right)\right). \tag{4.36}$$

As the monomials $x^{\boldsymbol{\alpha}} = \prod_{m=1}^{n}(x^{(m)})^{\alpha_m}$ with $|\boldsymbol{\alpha}| \leq k$ form a basis for $\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$, so will the set of elements which are units times these monomials. Thus, the image of $d\lambda_{i_j}$ contains a basis and hence $d\lambda_{i_j}$ is surjective.

For ii), suppose $u = \phi_i(x_{i_j})$ for some value of $j$. For $j' \neq j$, $\lambda_{i_{j'}}$ is a submersion by i). Thus, it is enough to consider the map

$$\lambda_{i_j} : V_{i_j} \to \mathbb{R} \times J^k(n,1). \tag{4.37}$$

Then $u^{(l)} = 0$ for all $l = 1, \ldots, n+1$ since $\phi_i(x_{i_j})$ is the origin in its local coordinate system. The derivative calculation in (4.36) becomes

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha},l}}}\left(j_1^k||\Phi_i(\cdot,\boldsymbol{v}) - u||^2(x)\right)\bigg|_{\boldsymbol{v}=0} = 2\left(\left(j_1^{k-|\boldsymbol{\alpha}|}\phi_i(x^{(l)})\right)\right)\left(\prod_{m=1}^{n+1}(x^{(m)})^{\alpha_m}\right)\right) \tag{4.38}$$

$$= 2x^{(l)} \cdot \prod_{m=1}^{n+1}(x^{(m)})^{\alpha_m} \tag{4.39}$$

for $l < n+1$. The monomials $\{x^{(l)} \cdot x^{\boldsymbol{\alpha}} : |\boldsymbol{\alpha}| \leq k, 1 \leq l \leq n\}$ will span $\mathfrak{m}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$, so $d\lambda_{i_j}$ is a submersion onto the subspace $\mathfrak{m}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$ of $\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$. Thus, $d\lambda_i$ maps onto $(\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1})^{\ell_i - 1} \oplus \mathfrak{m}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$. The fact that $W$ is assumed to be invariant under addition of constants means that, if $(w_1, \ldots, w_{\ell_i}) \in W$ and $c \in \mathbb{R}$, then $(w_1 + c, \ldots, w_{\ell_i} + c) \in W$. Therefore, the element $(1, \ldots, 1) \in TW$ and thus $\lambda_i$ is transverse to $W$. $\square$

**4.7.2. Transversality in the multi-distance function case.** In this section, we shall prove that $\Psi$ is transverse to any $W \subset {}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ with the ${}_{\boldsymbol{\ell}}\mathcal{R}^+-$invariance property, which will complete the proof of Theorem 4.3.1.

**Proposition 4.7.2.** *With $K$ as in (4.27), the mapping*

$$
\begin{aligned}
(4.40)\qquad & {}_{\boldsymbol{\ell}}j_1^k(\Gamma) : X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \times \mathcal{V} \to {}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2), \\
& (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) \mapsto ({}_{\ell_1}j_1^k \left( ||\Phi_1(\,\cdot\,, \boldsymbol{v}_1) - u_0||^2, ||\Phi_1(\,\cdot\,, \boldsymbol{v}_1) - u_1||^2 \right) (\boldsymbol{x}_1), \ldots, \\
& \qquad {}_{\ell_q}j_1^k \left( ||\Phi_q(\,\cdot\,, \boldsymbol{v}_q) - u_0||^2, ||\Phi_q(\,\cdot\,, \boldsymbol{v}_q) - u_q||^2 \right) (\boldsymbol{x}_q))
\end{aligned}
$$

*is transverse at $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{0}) \in K \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \times \mathcal{V}$ to any submanifold $W$ of ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ that is ${}_{\boldsymbol{\ell}}\mathcal{R}^+-$invariant.*

Proof. Let $U = \prod_{i=1}^{q} U_i = \prod_{i=1}^{q} \prod_{j=1}^{\ell_i} U_{i_j} \subset K_i$ with $U_{i_j} \subset K_{i_j}$ a coordinate neighborhood of $x_{i_j}$ for all $j = 1, \ldots, \ell_i$, and let $\mathcal{V} = \prod_{i=1}^{q} \mathcal{V}_i = \prod_{i=1}^{q} \prod_{j=1}^{\ell_i} V_{i_j}$. Fix $u_0 \in \mathbb{R}_0^{n+1}$ and restrict $(\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$ to $\{u_0\} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell}))}$. Let $\mathcal{Z} = (\mathcal{Z}_1 \times \ldots \times \mathcal{Z}_q) \subset (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell}))}$ be an open neighborhood of $(u_1, \ldots, u_q)$, with $u_i \in \mathcal{Z}_i$ for $i = 1, \ldots, q$. Locally, ${}_{\boldsymbol{\ell}}j_1^k(\Gamma)$ is given by

$$
(4.41)\qquad {}_{\boldsymbol{\ell}}j_1^k(\Gamma) : U \times \{u_0\} \times \mathcal{Z} \times \mathcal{V} \to U \times \prod_{i=1}^{q} (\mathbb{R}^2 \times J^k(n, 2))^{\ell_i},
$$

and it is therefore a product mapping. For each $i = 1, \ldots, q$, let

$$
\begin{aligned}
(4.42)\qquad & \beta_i : U_i \times \{u_0\} \times \mathcal{Z}_i \times \mathcal{V}_i \to U_i \times (\mathbb{R}^2 \times J^k(n, 2))^{\ell_i}, \\
& (\boldsymbol{x}_i, u_0, u_i, \boldsymbol{v}_i) \mapsto {}_{\ell_i}j_1^k(||\Phi_i(\,\cdot\,, \boldsymbol{v}_i) - u_0||^2, ||\Phi_i(\,\cdot\,, \boldsymbol{v}_i) - u_i||^2)(\boldsymbol{x}_i).
\end{aligned}
$$

We must show that $\beta_i$ is transverse to $W$ at $(\boldsymbol{x}_i, u_0, u_i, \mathbf{0})$. We may further restrict $\beta_i$ to each factor

$$\beta_{i_j} : U_{i_j} \times \{(u_0, u_i)\} \times V_{i_j} \to U_{i_j} \times \mathbb{R}^2 \times J^k(n,2), \tag{4.43}$$
$$(x,\, u_0,\, u_i,\, \boldsymbol{v}_{i_j}) \mapsto j_1^k \left( ||\widetilde{\Phi}_{i_j}(\cdot, \boldsymbol{v}_{i_j}) - u_0||^2, ||\widetilde{\Phi}_{i_j}(\cdot, \boldsymbol{v}_{i_j}) - u_i||^2 \right)(x)$$

for $j = 1, \ldots, \ell_i$. We represent the tangent space to $\mathbb{R}^2 \times J^k(n,2)$ algebraically as

$$\left(\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}\right)^{(2)}$$

with $\mathbf{x} = (x^{(1)}, \ldots, x^{(n)})$. The projection of $d\beta_{i_j}$ onto $TU_{i_j}$ is the identity map. At the point $(x_{i_j}, u_0, u_i, \mathbf{0}) \in U_{i_j} \times \{(u_0, u_i)\} \times V_{i_j}$, $\beta_{i_j}$ is equal to

$$j_1^k \left( ||\phi_i - u_0||^2, ||\phi_i - u_i||^2 \right)(x_{i_j}) = j_1^k \left(\sigma(\cdot, u_0), \sigma(\cdot, u_i)\right)(x_{i_j}). \tag{4.44}$$

The partial derivatives of $\beta_{i_j}$ with respect to the parameters in $V_{i_j}$ were calculated in Proposition 4.6.2; namely, for $l = 1, \ldots, n+1$,

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha},l}}}(\beta_{i_j})\bigg|_{\boldsymbol{v}=0} = \left( 2\left( j_1^{k-|\boldsymbol{\alpha}|}\phi_i^{(l)}(x) - u_0^{(l)} \right) \left( \prod_{m=1}^{n} (x^{(m)})^{\alpha_m} \right), \right. \tag{4.45}$$
$$\left. 2\left( j_1^{k-|\boldsymbol{\alpha}|}\phi_i^{(l)}(x) - u_i^{(l)} \right) \left( \prod_{m=1}^{n} (x^{(m)})^{\alpha_m} \right) \right)$$

for varying choices of $\boldsymbol{\alpha}$ with $\alpha_{n+1} = 0$.

Then there are two cases for each $i$: (1) $\phi_i(x_{i_j}) \neq u_0, u_i$ for any $j = 1, \ldots, \ell_i$, or (2) $u_0$ or $u_i$ equals $\phi_i(x_{i_j})$ for some $j$ (with the other possibly equal to $\phi_i(x_{i_k})$ for some $k \neq j$ since $u_0$ and $u_i$ are assumed to be distinct). Furthermore, each of these cases have several subcases.

- Case 1: We suppose that, for all $j = 1, \ldots, \ell_i$, $u_0 \neq \phi_i(x_{i_j})$ and $u_i \neq \phi_i(x_{i_j})$. There are three subcases: (i) the distance squared functions $\sigma(\cdot, u_0)$ and $\sigma(\cdot, u_i)$ are nonsingular at the origin in the local coordinate system about $x_{i_j}$, (ii) one of $\sigma(\cdot, u_0)$ and $\sigma(\cdot, u_i)$ is nonsingular and the other is singular, or (iii) both $\sigma(\cdot, u_0)$

and $\sigma(\cdot, u_i)$ are singular. By expanding (4.24) and taking partial derivatives, one determines that the condition for $\sigma(\cdot, u_0)$ to be singular at the origin is that

$$u_0^{(1)} = \ldots = u_0^{(n)} = 0, \tag{4.46}$$

and similarly for $\sigma(\cdot, u_i)$ and $u_i$. Geometrically, this corresponds to $u_0$ and $u_i$ lying on the normal line $L_{i_j}$ to $\mathcal{B}_i$ at the point $x_{i_j}$. In the local Monge coordinates, $L_{i_j}$ is the $x^{(n+1)}$–axis.

For (i), the nonsingularity of $\sigma(\cdot, u_0)$ and $\sigma(\cdot, u_i)$ means that $\beta_{i_j}(x_{i_j}, u_0, u_i, \mathbf{0})$ belongs to the open orbit of jets of nonsingular germs in the fiber $\mathbb{R}^2 \times J^k(n, 2)$ under the right equivalence action. By definition, $W$ is $\boldsymbol{\ell}\mathcal{R}^+$−invariant; hence, it contains an open subset containing $j_1^k\left(\sigma(\cdot, u_0), \sigma(\cdot, u_i)\right)(x_{i_j})$. In this case, $\beta_{i_j}$ is transverse to $W$.

For (ii), suppose for example that $\sigma(\cdot, u_0)$ is nonsingular but $\sigma(\cdot, u_i)$ is singular at $x_{i_j}$. Therefore, $u_i$ lies on the normal line $L_{i_j}$, so that $u_i^{(l)} = 0$ for $1 \leq l \leq n$ and $u_i^{(n+1)} \neq 0$, and $u_0 \notin L_{i_j}$. So, there exists $l$ such that $(u_0^{(l)}, u_0^{(n+1)})$ and $(u_i^{(l)}, u_i^{(n+1)})$ are linearly independent ($u_i^{(n+1)}$ and $u_0^{(l)}$ are nonzero). Thus, using (4.45), $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha},l}}}(\beta_{i_j})$ and $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha},n+1}}}(\beta_{i_j})$ span $\left(\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}\right)^{(2)}$ by varying $\boldsymbol{\alpha}$, i.e., $\beta_{i_j}$ is a submersion in this case.

Finally, for (iii), suppose $\sigma(\cdot, u_0)$ and $\sigma(\cdot, u_i)$ are both singular at the origin. Then $u_0, u_i \in L_{i_j}$, so $u_0^{(l)} = u_i^{(l)} = 0$ for $1 \leq l \leq n$ and $u_0^{(n+1)}$ and $u_i^{(n+1)}$ are distinct and nonzero. Due to the fact that $\sigma(\cdot, u_0)$ and $\sigma(\cdot, u_i)$ are singular, for $l = 1, \ldots, n$, (4.45) becomes

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha},l}}}(\beta_{i_j})\bigg|_{\boldsymbol{v}=0} = \left(2\left(x^{(l)}\right)\left(\prod_{m=1}^{n}(x^{(m)})^{\alpha_m}\right), 2\left(x^{(l)}\right)\left(\prod_{m=1}^{n}(x^{(m)})^{\alpha_m}\right)\right), \tag{4.47}$$

and for $l = n+1$, (4.45) equals

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},n+1}}(\beta_{i_j})\bigg|_{\boldsymbol{v}=0} = \left(2\left(j_1^{k-|\boldsymbol{\alpha}|} f_i(x^{(1)},\ldots,x^{(n)}) - u_0^{(n+1)}\right)\left(\prod_{m=1}^{n}(x^{(m)})^{\alpha_m}\right),\right.$$
$$\left. 2\left(j_1^{k-|\boldsymbol{\alpha}|} f_i(x^{(1)},\ldots,x^{(n)}) - u_i^{(n+1)}\right)\left(\prod_{m=1}^{n}(x^{(m)})^{\alpha_m}\right)\right), \tag{4.48}$$

where $j_1^{k-|\boldsymbol{\alpha}|} f_i(x^{(1)},\ldots,x^{(n)}) - u_0^{(n+1)}$ and $j_1^{k-|\boldsymbol{\alpha}|} f_i(x^{(1)},\ldots,x^{(n)}) - u_i^{(n+1)}$ are units in $\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}$. Thus, by varying $\boldsymbol{\alpha}$, $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\beta_{i_j})$ for $1 \le l \le n$ and $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},n+1}}(\beta_{i_j})$ span

$$\langle (u_0^{(n+1)}, u_i^{(n+1)}) \rangle \oplus \left(\mathfrak{m}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}\right)^{(2)}. \tag{4.49}$$

As $W$ is ${}_{\boldsymbol{\ell}}\mathcal{R}^{+}-$invariant, the element $(1,1) \in TW$. Then, since $(1,1)$ and $(u_0^{(n+1)}, u_i^{(n+1)})$ are linearly independent, we conclude that $\beta_{i_j}$ is transverse to $W$.

- Case 2: Suppose, for example, that $\phi_i(x_{i_j}) = u_i$ $(\neq u_0)$. Since $u_i$ is the origin in the local Monge coordinate system, $u_i^{(1)} = \ldots = u_i^{(n+1)} = 0$, and the condition in (4.46) implies that $\sigma(\cdot, u_i)$ has a critical point at the origin. From (4.23) and (4.24), the second derivative of $\sigma(\cdot, u_i)$ is degenerate provided that $u_i^{(n+1)} = 1/\kappa_j$ for some principal curvature $\kappa_j$ of $\mathcal{B}_i$ at the origin. Since $u_i^{(n+1)} = 0$, $\sigma(\cdot, u_i)$ has a nondegenerate Morse singularity at the origin.

There are two subcases: (a) $u_0 \notin L_{i_j}$, or (b) $u_0 \in L_{i_j}$. For both subcases, instead of using the fact that the projection of $d\beta_{i_j}$ onto $TU_{i_j}$ is the identity mapping, we shall utilize the fact that $TW$ contains $TU_{i_j}$. This enables us to utilize the partial derivatives of $\beta_{i_j}$ with respect to the surface coordinates in order to establish transversality of $\beta_{i_j}$ to $W$.

For (a), we know there is at least one $l \in \{1,\ldots,n\}$ such that $u_0^{(l)}$ is nonzero and

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\beta_{i_j})\bigg|_{\boldsymbol{v}=0} = \left(2\left(x^{(l)} - u_0^{(l)}\right)\left(\prod_{m=1}^{n}(x^{(m)})^{\alpha_m}\right), 2\left(x^{(l)}\right)\left(\prod_{m=1}^{n}(x^{(m)})^{\alpha_m}\right)\right). \tag{4.50}$$

The element $(1,1) \in TW$; moreover, the invariance of $W$ implies that $W$ contains an open subset containing $j_1^k\left(\sigma(\cdot, u_0), \sigma(\cdot, u_i)\right)(x_{i_j})$, where $\sigma(\cdot, u_i)$ has a Morse singularity at the origin. Thus, by varying $\boldsymbol{\alpha}$ in (4.50), $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\beta_{i_j})$ and $TW$ span

$$\langle (u_0^{(l)}, 0)\rangle \oplus \langle (1,1)\rangle \oplus (\mathfrak{m}_{\boldsymbol{x}}^2/\mathfrak{m}_{\boldsymbol{x}}^{k+1})^{(2)}.$$

Now, the partial derivative of $\beta_{i_j}$ with respect to each surface coordinate $x^{(p)}$ was calculated in Proposition 4.6.2:

$$\frac{\partial}{\partial x^{(p)}}(\beta_{i_j})\bigg|_{\boldsymbol{v}=0} = \left(2(x^{(p)} - u_0^{(p)}) + 2j_1^k\left((f_i - u_0^{(n+1)}) \cdot \frac{\partial f_i}{\partial x^{(p)}}\right), 2x^{(p)} + 2j_1^k\left(f_i \cdot \frac{\partial f_i}{\partial x^{(p)}}\right)\right).$$

Then, from $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\beta_{i_j})$ for varying $\boldsymbol{\alpha}$ and $\dfrac{\partial}{\partial x^{(p)}}(\beta_{i_j})$ for $p = 1, \ldots, n$, we may obtain all basis elements of $\left(\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}\right)^{(2)}$ of the form $(x^{(p)}, 0), (0, x^{(p)})$. Therefore, we conclude that the derivatives $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\beta_{i_j})$ and $\dfrac{\partial}{\partial x^{(p)}}(\beta_{i_j})$, together with $TW$, span $\left(\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}\right)^{(2)}$.

Finally, for (b), since $\sigma(\cdot, u_0)$ is singular at the origin, $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\beta_{i_j})$ for $1 \leq l \leq n$ is of the form in (4.47) and $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},n+1}}(\beta_{i_j})$ is of the form

$$\frac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},n+1}}(\beta_{i_j})\bigg|_{\boldsymbol{v}=0} = \Bigg(2\left(j_1^{k-|\boldsymbol{\alpha}|} f_i(x^{(1)}, \ldots, x^{(n)}) - u_0^{(n+1)}\right)\left(\prod_{m=1}^{n} (x^{(m)})^{\alpha_m}\right), \\ 2\left(j_1^{k-|\boldsymbol{\alpha}|} f_i(x^{(1)}, \ldots, x^{(n)})\right)\left(\prod_{m=1}^{n} (x^{(m)})^{\alpha_m}\right)\Bigg), \tag{4.51}$$

where $u_0^{(n+1)} \neq 0$. As in case (a), the partial derivatives $\dfrac{\partial}{\partial v_{j_{\boldsymbol{\alpha}},l}}(\beta_{i_j})$ for varying choices of $\boldsymbol{\alpha}$, and $\dfrac{\partial}{\partial x^{(p)}}(\beta_{i_j})$ for $p = 1, \ldots, n$, together with $TW$, span $\left(\mathcal{E}_{\boldsymbol{x}}/\mathfrak{m}_{\boldsymbol{x}}^{k+1}\right)^{(2)}$.

$\square$

Finally, we are able to apply the Hybrid Multi-Transversality Theorem to conclude that, given any ${}_{\boldsymbol{\ell}}\mathcal{R}^{+}-$invariant Whitney stratified submanifold $W$ of ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$,

$$\mathscr{W} = \{\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1}) : {}_{\boldsymbol{\ell}}j_1^k\rho_\phi \text{ is transverse on } Z \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)} \text{ to } W \text{ in } {}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)\}$$

is an open dense subset provided that $Z \subset X^{(\boldsymbol{\ell})}$ is compact, and is a residual subset otherwise. This proves Theorem 4.3.1. $\square$

CHAPTER 5

# Classification Theorems for Generic Linking

## 5.1. Introduction

In this chapter, we use the transversality theorem proven in Chapter 4 to present a classification of generic linking in $\mathbb{R}^{n+1}$ for $n \leq 6$. In Section 5.2, we introduce the submanifolds to which we shall apply the transversality theorem in order to establish the classification. This will include all submanifolds of jet space corresponding to linking configurations that, we shall prove, will generically occur, as well as a finite list of closed Whitney stratified subsets of jet space representing all configurations that will be generically avoided for codimension reasons.

Then, in Section 5.3, we explain the consequences of Theorem 4.3.1 applied to the submanifolds in Section 5.2, enabling us to prove the generic linking classification theorems for $n = 1, 2$ (Theorems 3.4.3 and 3.4.4). We also present the classification results in dimensions $3 \leq n \leq 6$ in a series of tables in Section 5.5. In Section 5.4, we prove Theorems 3.5.3, 3.5.6, and 3.5.7 from Chapter 3 regarding the refined stratifications of the boundaries and medial axes.

## 5.2. Submanifolds for the transversality theorem

In this section, we introduce the classes of submanifolds and Whitney stratified sets of the multijet space ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ to which we will apply the transversality theorem of Chapter 4. We shall denote this set of submanifolds and stratified sets by ${}_{\boldsymbol{\ell}}\mathcal{S}$. The submanifolds in ${}_{\boldsymbol{\ell}}\mathcal{S}$ will be one of four types: (i) those formed from simple multigerms; (ii) those formed from the partial multijet orbits from Chapter 4; (iii) those which characterize geometric features of boundary points for self–linking; and (iv) those of higher codimension which arise as strata of closed Whitney stratified sets.

We begin by recalling the correspondence between submanifolds of jet space and singularities of smooth functions as orbits of a group action, which we use to determine the form that the submanifolds will take to which we shall apply Theorem 4.3.1.

**5.2.1. Submanifolds for a single distance function.** Recall from Section 1.3 in Chapter 1 that, if $N$ is a smooth manifold, the group of $k$–jets of diffeomorphism germs $N \to N$ acts on the jet space fiber $J^k(N, \mathbb{R})_x \cong J^k(n, 1) \times \mathbb{R}$ for $x \in N$. Since the group action is algebraic, the orbits under the algebraic group action are submanifolds of the fiber [**27**]. These are $\mathcal{R}$−orbits which form a subbundle of $J^k(N, \mathbb{R})$. Similarly, if $S$ denotes a set of $r$ distinct points in $N$, we obtain a corresponding action of ${}_r\mathcal{R}^+$ on the multijet space fiber ${}_rJ^k(N, \mathbb{R})_S \cong (J^k(n, 1) \times \mathbb{R})^r$. These multi–orbits were needed for Mather's classification theorem (Theorem 1.4.4 in Chapter 1).

Assume that $k \geq 9$, and let $\mathscr{W}^j$ denote the orbit in $J^k(X, \mathbb{R})_x$ of a germ $(X, x) \to \mathbb{R}$ with an $A_j$ singularity at $x$. The following finite list gives the specific submanifolds of $J^k(X, \mathbb{R})_x$ needed for the generic classification of the local normal forms of the medial axis in dimensions $n \leq 6$:

- $\mathscr{W}^1$ (denoted by $W_0^A$ in [**32**]),
- $\mathscr{W}^3$ (denoted by $W_2^B$ in [**32**]),
- $\mathscr{W}^5$ (denoted by $W_4^C$ in [**32**]), and
- $\mathscr{W}^7$ (denoted by $W_6^D$ in [**32**]).

Let $W^j$ denote the subbundle of $J^k(X, \mathbb{R})$ with fiber at $x \in X$ equal to $\mathscr{W}^j$.

Let $\eta = c_1\mathscr{W}^1, c_3\mathscr{W}^3, c_5\mathscr{W}^5, c_7\mathscr{W}^7$ denote a finite number of choices of the orbits above for some nonnegative integer coefficients $c_j$, where the $c_j$ need not be distinct. If $|\eta| = r = \sum_j c_j$, let $\mathscr{W}^\eta$ denote the submanifold of ${}_rJ^k(X, \mathbb{R})_S$ consisting of elements $(z_1, \ldots, z_r)$ such that exactly $c_i$ of the jets are in $\mathscr{W}^i$ for all $i$ and such that all target values of the $z_i$'s are equal. In our usual notation, $\mathscr{W}^\eta$ corresponds to the orbit of a multigerm of type $\mathcal{A}_\eta$. Let $W^\eta$ denote the subbundle of ${}_rJ^k(X, \mathbb{R})$ over $X^{(r)}$ formed from the orbits $\mathscr{W}^\eta$ in the fibers.

**5.2.2. Distinguished class of submanifolds in ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ corresponding to linking type.** Next, we define the submanifold of the partial multijet space ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ associated to the linking configuration $(\mathcal{A}_\alpha : \mathcal{A}_{\beta_1}, \ldots, \mathcal{A}_{\beta_q})$ for the multi–distance function. This involves taking products of the multi–orbits in the previous section.

As in Chapter 4, let $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$ for nonnegative integers $\ell_i$, and let $q(\boldsymbol{\ell})$ denote the number of nonzero $\ell_i$. Lemma 4.2.4 in Chapter 4 showed that ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ is a locally trivial fiber bundle with fiber at $S = (S_1, \ldots, S_q)$ in $X^{(\boldsymbol{\ell})}$ equal to

$$\prod_{i=1}^{q} {}_{\ell_i}J^k(X_i, \mathbb{R}^2)_{S_i} \cong \prod_{i=1}^{q} \left(J^k(n, 2) \times \mathbb{R}^2\right)^{\ell_i}. \tag{5.1}$$

For the linking configuration $(\mathcal{A}_\alpha : \mathcal{A}_{\beta_1}, \ldots, \mathcal{A}_{\beta_q})$, consider the corresponding multi–orbits $\mathscr{W}^{\beta_i}$ in the fibers ${}_{\ell_i}J^k(X_i, \mathbb{R})_{S_i}$ for $i = 1, \ldots, q$. As in the multijet case, the multi–orbits $\mathscr{W}^{\beta_i}$ in the fibers over points in $X_i^{(\ell_i)}$ fit together to form subbundles $W^{\beta_i}$ in ${}_{\ell_i}J^k(X_i, \mathbb{R})$ for $i = 1, \ldots, q$. Then $W^{\beta_1} \times \ldots \times W^{\beta_q} \subset \prod_{i=1}^{q} {}_{\ell_i}J^k(X_i, \mathbb{R})$ is a subbundle over $X^{(\boldsymbol{\ell})}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_q)$ so that for $i = 1, \ldots, q$, $W^{\alpha_i}$ denotes the subbundle of $J^k(X, \mathbb{R})$ over $X_i$ with fiber at $x$ equal to $\mathscr{W}^{\alpha_i}$, the orbit of a germ with an $A_{\alpha_i}$ singularity. Let

$$W^\alpha = (W^{\alpha_1} \times \underbrace{J^k(X_1, \mathbb{R}) \times \ldots \times J^k(X_1, \mathbb{R})}_{\ell_1 - 1}) \times \ldots \times (W^{\alpha_q} \times \underbrace{J^k(X_q, \mathbb{R}) \times \ldots \times J^k(X_q, \mathbb{R})}_{\ell_q - 1}),$$

where for each $i$, $\left(W^{\alpha_i} \times J^k(X_i, \mathbb{R}) \times \ldots \times J^k(X_i, \mathbb{R})\right)$ is restricted to lie over $X_i^{(\ell_i)}$. Therefore, $W^\alpha \subset \prod_{i=1}^{q} {}_{\ell_i}J^k(X_i, \mathbb{R})$ is a fiber bundle over $X^{(\boldsymbol{\ell})}$, and

$$W^\alpha \times W^{\beta_1} \times \ldots \times W^{\beta_q} \subset \prod_{i=1}^{q} {}_{\ell_i}J^k(X_i, \mathbb{R}) \times \prod_{i=1}^{q} {}_{\ell_i}J^k(X_i, \mathbb{R}).$$

Then, let $W^{(\alpha:\boldsymbol{\beta})}$ denote the restriction of the bundle $W^\alpha \times W^{\beta_1} \times \ldots \times W^{\beta_q}$ to lie over $X^{(\boldsymbol{\ell})}$. $W^{(\alpha:\boldsymbol{\beta})}$ is a smooth submanifold of ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$, and it is the submanifold associated to the linking configuration $(\mathcal{A}_{\boldsymbol{\alpha}} : \mathcal{A}_{\boldsymbol{\beta}_1}, \ldots, \mathcal{A}_{\boldsymbol{\beta}_q})$.

**5.2.3. Submanifolds capturing geometric properties of the boundaries.** We next consider certain cases for self–linking, which depend on the differential geometry of the hypersurface $\phi_i(X_i) = \mathcal{B}_i$. To do so, we return to the Monge representation as in Section 4.6 in Chapter 4. If $x^{(1)}, \ldots, x^{(n)}$ denote local coordinates on $X_i$ and $\kappa_1, \ldots, \kappa_n$ denote the principal curvatures of $\mathcal{B}_i$ at the origin, then we may locally write $\mathcal{B}_i$ in Monge form as $(x^{(1)}, \ldots, x^{(n)}, f(x^{(1)}, \ldots, x^{(n)}))$, where

$$f(x^{(1)}, \ldots, x^{(n)})) = \sum_{i=1}^{n} \frac{1}{2}\kappa_i (x^{(i)})^2 + \sum_{|\boldsymbol{\alpha}| \geq 3} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}.$$

Furthermore, we assume that the coordinates are chosen so that $\kappa_1 > \kappa_2 > \ldots > \kappa_n$ and $\kappa_n < 0$. We consider the case where the distance squared function to a point $u_0$ has a critical point, so $u_0^{(i)} = 0$ for $i < n+1$. In other words, $u_0$ lies on the normal line to the surface at the origin. Then the distance squared function to $u_0$ is given by

$$\begin{aligned}
\sigma(\cdot, u_0) &= \sum_{i=1}^{n} (x^{(i)})^2 + \left( \sum_{i=1}^{n} \frac{1}{2}\kappa_i (x^{(i)})^2 + \sum_{|\boldsymbol{\alpha}| \geq 3} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} - u_0^{(n+1)} \right)^2 \\
&= \sum_{i=1}^{n} \left(1 - u_0^{(n+1)}\kappa_i\right)(x^{(i)})^2 - 2u_0^{(n+1)} \sum_{|\boldsymbol{\alpha}| \geq 3} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} + (u_0^{(n+1)})^2 \\
&\qquad + \left( \sum_{i=1}^{n} \frac{1}{2}\kappa_i (x^{(i)})^2 + \sum_{|\boldsymbol{\alpha}| \geq 3} a_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} \right)^2 .
\end{aligned} \tag{5.2}$$

If (i) $u_0^{(n+1)} = 1/\kappa_1$, and (ii) if the coefficient of the monomial $(x^{(1)})^3$ is 0, so that every quadratic and cubic term of the distance squared function is divisible by one or more of the monomials $x^{(2)}, \ldots, x^{(n)}$, then the distance squared function to $u_0$ has an $A_3$ singularity at the origin. This is because, after completing the square to eliminate terms of degree 3 and 4 in $x^{(2)}, \ldots, x^{(n)}$, one can change coordinates so that the 4–jet of the distance squared function is of the form $(x^{(1)})^4 + \sum_{i=2}^{n} (x^{(i)})^2$. Since $u_0^{(n+1)} = 1/\kappa_1$, $1 - u_0^{(n+1)}\kappa_i > 0$ for all $i > 1$ because $u_0^{(n+1)}\kappa_i < 1$ by the way the coordinates were chosen. Thus, the $A_3$ singularity is a local minimum of the distance squared function.

For the $(A_3 : A_3)$ linking type, the distance squared function to $u_0$ and the distance squared function to a point $u_i \neq u_0$ both have an $A_3$ singularity at the origin. This translates into the following conditions: (1) $u_0 = (0, \ldots, 0, u_0^{(n+1)})$ and $u_i = (0, \ldots, 0, u_i^{(n+1)})$;

(2) $u_0^{(n+1)} = 1/\kappa_1$ and $u_i^{(n+1)} = 1/\kappa_n$, with $u_0^{(n+1)}$ and $u_i^{(n+1)}$ having opposite signs so that $u_0$ and $u_i$ are on opposite sides of the hypersurface; and (3) the coefficients of the monomials $(x^{(1)})^3$ and $(x^{(n)})^3$ are both 0. Now, (3) places two conditions on the hypersurface $\mathcal{B}_i$.

In $\mathbb{R}^3$, the condition that the distance squared functions from points $u_0$ and $u_i$ both have an $A_3$ singularity at the origin implies that the origin belongs to two distinct crest curves on $\mathcal{B}_i$. (Recall from the description of the $A_3$ case in Section 1.5.3 in Chapter 1 that a crest curve consists of points satisfying the property that the larger principal curvature in absolute value at the point is a maximum along the associated principal direction.) The two principal curvatures $\kappa_1$ and $\kappa_2$, which are of opposite signs, must be equal in absolute value as they are both maxima along their associated principal directions. This means the surface $\mathcal{B}_i$ is locally a saddle. Let $x = x^{(1)}, y = x^{(2)}$. A direct calculation shows that the tangent to the crest line corresponding to $\kappa_2$ is along the principal direction $x = 0$ [**9**]. Analogously, the tangent to the other crest line corresponding to $\kappa_1$ is along the principal direction $y = 0$. This implies that the two crest curves are transverse on the surface $\mathcal{B}_i$. The submanifold representing the $(A_3 : A_3)$ configuration is a codimension 2 submanifold.

**Remark 5.2.1.** Note that if the origin were an umbilic point in $\mathbb{R}^3$, the distance squared function from any point $u$ could not have an $A_3$ singularity at the origin. For, if $u^{(3)} = 1/\kappa_1 = 1/\kappa_2$, the Hessian of the distance squared function would have rank 0.

In this thesis, we do not include the details of the corresponding transversality calculations for the $(A_3 : A_3)$ case in dimensions $n \geq 3$, so the genericity of the $(A_3 : A_3)$ linking type is only proven in $\mathbb{R}^3$. Other self–linking configurations that involve the differential geometric properties of the hypersurface include the configurations $(A_3 : A_5)$, $(A_5 : A_3)$, $(A_3 : A_7)$, $(A_7 : A_3)$, and $(A_5 : A_5)$. We do not include the calculations for the specific conditions on the differential geometric properties of the surfaces in these cases, or the proofs that they are generic. Nevertheless, we do expect all of these self–linking configurations involving higher order $A_k$ singularities to be generic.

**5.2.4. Closed stratified sets of higher codimension.** The third class of submanifolds arise as strata of a finite list of closed Whitney stratified sets which will include all linking configurations that will be shown to be non–generic. Within this list, there are three types of strata: (1) submanifolds for simple multigerms; (2) those in the closure of a submanifold $W^{(\alpha:\boldsymbol{\beta})}$; and (3) those representing the degeneracy of the geometric conditions on the surface.

For (1), in his classification of generic medial axis strata for $n \leq 6$, Mather gave the following three submanifolds of $J^k(X,\mathbb{R})_x$ corresponding to strata of codimension $> 7$:

- $\mathscr{W}^9$ (denoted by $W_8^E$ in [**32**], and see below),
- $\mathscr{W}_7^F = \left\{ z \in J^k(X,\mathbb{R})_x : \exists \mu > 0, \mu \neq 1 \text{ with } z = (x_1^2+x_2^2)(x_1^2+\mu x_2^2) + \sum_{i=3}^{n} x_i^2 \right\}$,
- $\mathscr{W}_8^G = \left\{ z \in J^k(X,\mathbb{R})_x : z \notin W_7^F, j^3 z = \sum_{i=c+1}^{n} x_i^2, c \geq 2 \right\}$.

Note that $\mathscr{W}^9$ denotes the orbit in $J^k(X,\mathbb{R})_x$ of a germ $(X,x) \to \mathbb{R}$ with an $A_9$ singularity at $x$, and we let $W^9$ denote the corresponding subbundle of $J^k(X,\mathbb{R})$. Then the closure of the smooth submanifold $W^9$ is the closed Boardman stratum $\overline{\Sigma^{n,1_8}}$, which was introduced in Section 1.3 in Chapter 1. The submanifold $\mathscr{W}_7^F$ is invariant under the $\mathcal{R}-$action, whereas $\mathscr{W}_8^G$ is a finite union of submanifolds invariant under the group action. $\mathscr{W}_7^F$ and $\mathscr{W}_8^G$ have the property that their closures are semi–algebraic sets and thus are Whitney stratified. We denote by $W_7^F$ and $W_8^G$ the corresponding fiber bundles. Similarly, for sufficiently large $k$, we identify in $(J^k(n,1)\times\mathbb{R})^r$ the $k$–multijet orbits of families of multigerms having $\mathcal{R}_e^+-$codimension $\geq n+2$, which have the property that their closures are Whitney stratified sets. For such a multigerm $\mathcal{A}_\eta$, the closure of the multi–orbit $\mathscr{W}^\eta$ in ${}_rJ^k(X,\mathbb{R})_S \cong (J^k(n,1)\times\mathbb{R})^r$ is given for appropriate values of $j_k$ by the product

$$\overline{\mathscr{W}^\eta} := \overline{\mathscr{W}^{j_1}} \times \ldots \times \overline{\mathscr{W}^{j_r}} \times \Lambda\mathbb{R}^r, \tag{5.3}$$

where $\overline{\mathscr{W}^{j_1}} \subset J^k(n,1)$ and $\Lambda\mathbb{R}^r$ denotes the diagonal line in $\mathbb{R}^r$.

If $W^{\eta}$ denotes the subbundle of ${}_rJ^k(X, \mathbb{R})$ formed from the fibers $\mathscr{W}^{\eta}$ at $S$ in $X^{(r)}$, then $\overline{W^{\eta}}$ is a stratified set that will contain the $k$–multijet orbits of families of codimension $> n+1$. Then, transversality to the finite list of Whitney stratified sets establishes transversality to any other submanifold of higher codimension belonging to the closure of one of the submanifolds in the list.

For (2), we determine the closure of $W^{(\alpha:\boldsymbol{\beta})}$, denoted $\overline{W^{(\alpha:\boldsymbol{\beta})}}$, in ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$. Namely, $\overline{W^{(\alpha:\boldsymbol{\beta})}}$ is given by the product of the closures of the submanifolds that comprise $W^{(\alpha:\boldsymbol{\beta})}$:

$$\overline{W^{(\alpha:\boldsymbol{\beta})}} := \overline{W^{\alpha}} \times \overline{W^{\beta_1}} \times \ldots \times \overline{W^{\beta_q}}, \tag{5.4}$$

where the product on the right–hand side lies over $X^{(\boldsymbol{\ell})}$. This yields a finite list of closed Whitney stratified sets in ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$, as Mather's classification in the single distance function case implies that there are only a finite number of possibilities of multi–orbits to consider. In each case, transversality of ${}_{\boldsymbol{\ell}}j_1^k\rho_{\phi}$ to $W^{(\alpha:\boldsymbol{\beta})}$ will establish transversality to each stratum of higher codimension in its closure $\overline{W^{(\alpha:\boldsymbol{\beta})}}$.

For (3), we include the conditions in $\mathbb{R}^3$ that the distance squared functions from $u_0$ and $u_i$ have $A_4$ singularities at the origin. From [**21**], the condition that the distance squared function from $u_0$ has an $A_4$ singularity at the origin is

$$4b_1^2 + (\kappa_1 - \kappa_2)(8c_0 - \kappa_1^3) = 0,$$

where $b_1$ and $c_0$ are the coefficients of the monomials $x^2y$ and $x^4$, respectively. Similarly, from [**9**], the condition that the distance squared function from $u_i$ has an $A_4$ singularity at the origin is

$$4b_2^2 - (\kappa_1 - \kappa_2)(8c_4 - \kappa_2^3) = 0,$$

where $b_2$ and $c_4$ are the coefficients of the monomials $xy^2$ and $y^4$, respectively. This places two additional conditions on the surface $\mathcal{B}_i$, namely, on the fourth–order coefficients $c_0$ and $c_4$. Likewise, there are conditions for the distance squared functions to have $A_{\geq 5}$ singularities, although we do not include those calculations. We also point out that there are additional submanifolds that represent the degeneracy of the geometric conditions on a hypersurface for the self–linking configurations listed at the end of Section 5.2.3.

## 5.3. Consequences of transversality to elements of ${}_{\boldsymbol{\ell}}\mathcal{S}$

In this section, we consider the consequences of Theorem 4.3.1 applied to the three classes of submanifolds and stratified sets in ${}_{\boldsymbol{\ell}}\mathcal{S}$. Let $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ with $n \leq 6$, and let $\mathcal{P} \subset \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ consist of all $\phi \in \mathcal{P}$ such that ${}_{\boldsymbol{\ell}}j_1^k \rho_\phi$ is transverse to every element of ${}_{\boldsymbol{\ell}}\mathcal{S}$. The set $\mathcal{P}$ is a residual subset of $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ by Theorem 4.3.1.

For $\phi \in \mathcal{P}$, transversality of ${}_{\boldsymbol{\ell}}j_1^k \rho_\phi$ to $W^{(\alpha:\boldsymbol{\beta})}$ in ${}_{\boldsymbol{\ell}}\mathcal{S}$ yields transversality statements for certain submanifolds in the product space $X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$. In what follows, we show how transversality of these submanifolds in $X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$ implies transversality results for certain projections of the submanifolds, which are needed for the classification of generic linking.

Let $(\boldsymbol{x}, \boldsymbol{u}) = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_q, u_0, \ldots, u_q) \in X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$, where $\boldsymbol{x}_j = (x_{j_1}, \ldots, x_{j_{\ell_j}}) \in X_j^{(\ell_j)}$. For $j = 1, \ldots, q$, choose neighborhoods $U_j = \prod_{i=1}^{\ell_j} U_j^{(i)}$ where $x_{j_i} \in U_j^{(i)}$ for every $i$. Let $\widetilde{X} := \prod_{j=1}^{q} U_j \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$, and let

$$\pi_0 : \widetilde{X} \to \prod_{j=1}^{q} U_j \times \mathbb{R}_0^{n+1}$$

and, for $j = 1, \ldots, q$,

$$\pi_j : \widetilde{X} \to U_j \times \mathbb{R}_j^{n+1}$$

be projections. For $s = \sum_{j=1}^{q} \ell_j$, we have the jet extension mapping

$${}_s j_1^k \sigma_0(\cdot, u_0) : \prod_{j=1}^{q} U_j \times \mathbb{R}_0^{n+1} \to \prod_{j=1}^{q} {}_{\ell_j} J^k(X_j, \mathbb{R})$$

and for each $j$, we have

$${}_{\ell_j} j_1^k \sigma_j(\cdot, u_j) : U_j \times \mathbb{R}_j^{n+1} \to {}_{\ell_j} J^k(X_j, \mathbb{R}).$$

By Theorem 4.3.1, the mapping

$$(5.5) \qquad {}_{\boldsymbol{\ell}}j_1^k\rho_\phi : \widetilde{X} \to {}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$$

is transverse to $W^{(\alpha:\boldsymbol{\beta})}$ in ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$. By the way the mapping ${}_{\boldsymbol{\ell}}j_1^k\rho_\phi$ is defined (see (4.40) in Chapter 4), this is equivalent to the following:

(1) ${}_sj_1^k\sigma_0(\cdot, u_0) \circ \pi_0$ is transverse to $W^\alpha$;

(2) $\prod_j({}_{\ell_j}j_1^k\sigma_j(\cdot, u_j) \circ \pi_j)$ is transverse to $W^{\beta_1} \times \ldots \times W^{\beta_q}$, which holds if and only if each ${}_{\ell_j}j_1^k\sigma_j(\cdot, u_j)$ is transverse to $W^{\beta_j}$; and

(3) all of the $({}_{\ell_j}j_1^k\sigma_j(\cdot, u_j) \circ \pi_j)^{-1}(W^{\beta_j})$ are in general position and their intersection is transverse to $({}_sj_1^k\sigma_0(\cdot, u_0) \circ \pi_0)^{-1}(W^\alpha)$.

Now, (2) implies that the distance squared function families $\sigma_i, i = 1, \ldots, q$ involved in the configuration are versal unfoldings.

Let $\widetilde{U} = \prod_{j=1}^{q} U_j^{(1)}$, and consider the mapping

$${}_{q(\boldsymbol{\ell})}j_1^k\sigma_0(\cdot, u_0) : \widetilde{U} \times \mathbb{R}_0^{n+1} \to {}_{q(\boldsymbol{\ell})}J^k(X, \mathbb{R}).$$

With $\alpha = (\alpha_1, \ldots, \alpha_q)$, let $W_1^\alpha \subset {}_{q(\boldsymbol{\ell})}J^k(X, \mathbb{R})$ be the subbundle with fiber at $S_0$ in $\widetilde{U}$ equal to

$$(5.6) \qquad \mathscr{W}^{\alpha_1} \times \ldots \times \mathscr{W}^{\alpha_q} \times \Lambda\mathbb{R}^{q(\boldsymbol{\ell})}.$$

By the way $W^\alpha$ is defined, the transversality theorem implies that ${}_{q(\boldsymbol{\ell})}j_1^k(\sigma_0(\cdot, u_0))$ is transverse to $W_1^\alpha$. Thus, $\sigma_0$ is a versal unfolding of $\sigma_0(\cdot, u_0)$. Let

$$(5.7) \qquad \widetilde{W^\alpha} = {}_{q(\boldsymbol{\ell})}j_1^k\sigma_0(\cdot, u_0)^{-1}(W_1^\alpha) \subset \widetilde{U} \times \mathbb{R}_0^{n+1}$$

and for $j = 1, \ldots, q$, let

$$(5.8) \qquad \widetilde{W^{\beta_j}} = {}_{\ell_j}j_1^k\sigma_j(\cdot, u_j)^{-1}(W^{\beta_j}) \subset U_j \times \mathbb{R}_j^{n+1}.$$

Then we have the following spaces in $\widetilde{X}$ :

$$Z_0 := \widetilde{W^{\alpha}} \times \prod_{j=1}^{q} \left( \prod_{i=2}^{\ell_j} U_j^{(i)} \times \mathbb{R}_j^{n+1} \right), \tag{5.9}$$

$$Z_j := \widetilde{W^{\beta_j}} \times \prod_{i \neq j} U_i \times \prod_{i \neq j} \mathbb{R}_i^{n+1}, \quad j = 1, \dots, q.$$

Then, letting $\widetilde{W^{\boldsymbol{\beta}}} = \widetilde{W^{\beta_1}} \times \ldots \times \widetilde{W^{\beta_q}}$, we obtain

$$\bigcap_{j=1}^{q} Z_j = \widetilde{W^{\boldsymbol{\beta}}} \times \mathbb{R}_0^{n+1},$$

and the transversality theorem implies that $Z_0$ is transverse to $\bigcap_{j=1}^{q} Z_j$ in $\widetilde{X}$.

We now finish the proof that the linking configuration $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$ is generic. For a fixed $j \in \{1, \dots, q\}$, let $V_0 = \mathbb{R}_0^{n+1}$, $V_1 = \prod_{i=2}^{\ell_j} U_j^{(i)} \times \mathbb{R}_j^{n+1}$, and $V_2 = \prod_{k \neq j} \left( \prod_{i=2}^{\ell_k} U_k^{(i)} \times \mathbb{R}_k^{n+1} \right)$. Then $\widetilde{X} = \widetilde{U} \times V_0 \times V_1 \times V_2$. Note that $\widetilde{W^{\alpha}} \subset \widetilde{U} \times V_0$, $\widetilde{W^{\beta_j}} \subset \widetilde{U} \times V_1$, and that $Z_0$ is transverse to $Z_j$ in $\widetilde{X}$. Now, $\pi_{\widetilde{X}} : \widetilde{X} \to \widetilde{U}$ is a locally trivial fiber bundle. For every $j$, the projection of $Z_j$ under $\pi_{\widetilde{X}}$ is given by

$$\widetilde{W^{\beta_j}}_{\widetilde{U}} := \pi_{\widetilde{X}}(Z_j) = \Sigma'_{\beta_j} \times \prod_{i \neq j} U_i^{(1)}, \tag{5.10}$$

and the projection is a submersion (in fact, a diffeomorphism). Here, $\Sigma'_{\beta_j} \subset U_j^{(1)}$ is diffeomorphic to the stratum $\Sigma_{\beta_j}$ on $\mathcal{B}_j$ under the diffeomorphism between $X_j$ and $\mathcal{B}_j$ that the embedding $\phi$ provides. The projection of $Z_0$ onto $\widetilde{U}$ is also a submersion, and we let

$$\widetilde{W^{\alpha}}_{\widetilde{U}} = \pi_{\widetilde{X}}(Z_0) = \Sigma'_{\alpha}. \tag{5.11}$$

**Lemma 5.3.1.** *$\widetilde{W^{\beta_j}}_{\widetilde{U}}$ is transverse to $\widetilde{W^{\alpha}}_{\widetilde{U}}$ in $\widetilde{U}$.*

Proof. Let $w \in \widetilde{W^{\beta_j}}_{\widetilde{U}} \bigcap \widetilde{W^{\alpha}}_{\widetilde{U}}$. Then there exists $v_0 \in V_0$ and $v_1 \in V_1$ such that $(w, v_0) \in \widetilde{W^{\alpha}}_{\widetilde{U}} \times V_0$ and $(w, v_1) \in \widetilde{W^{\beta_j}}_{\widetilde{U}} \times V_1$. Then for any $v_2 \in V_2$, at the point

$(w, v_0, v_1, v_2) \in \widetilde{X}$, we have that

$$T(\widetilde{W^{\beta_j}} \times V_0 \times V_2) + T(\widetilde{W^{\alpha}} \times V_1 \times V_2) = T\widetilde{X}$$

by transversality of $Z_0$ and $Z_j$. Applying $d\pi_{\widetilde{X}}$ to the above equation, we obtain

$$T_w \widetilde{W^{\beta_j}}_{\widetilde{U}} + T_w \widetilde{W^{\alpha}}_{\widetilde{U}} = T_w \widetilde{U} \tag{5.12}$$

since the projections are submersions. □

By projecting both sides of equation (5.12), we obtain

$$T_w(\Sigma'_{\beta_j} \times \prod_{i \neq j} U_i^{(1)}) \times \widetilde{W^{\alpha}}_{\widetilde{U}} = T_w \widetilde{U}.$$

If $w = (w_1, \ldots, w_q)$, then after projecting the above equation onto the $U_j^{(1)}$ factor, we obtain

$$T_{w_j} \Sigma'_{\beta_j} \times T_{w_j} \Sigma'_{\alpha_j} = T_{w_j} U_j^{(1)}, \tag{5.13}$$

where $\Sigma'_{\alpha_j}$ is the projection of $\Sigma'_{\alpha}$ onto $T_{w_j} U_j^{(1)}$. Using the diffeomorphism that $\phi$ provides, Lemma 5.3.1 then implies that the strata $\Sigma_{\mathcal{A}_\alpha}$ and $\Sigma_{\mathcal{A}_{\beta_j}}$ intersect transversely in $\mathcal{B}$. This holds for all $j = 1, \ldots, q$, and we conclude that the linking configuration $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$ is generic.

Next, we prove that we obtain a Whitney stratification of $\widetilde{U}$ by linking type, which will imply that the refinement of the boundary stratification is Whitney. Let $V_3 = \prod_{j=1}^{q} \left( \prod_{i=2}^{\ell_j} U_j^{(i)} \times \mathbb{R}_j^{n+1} \right)$, so that $\widetilde{X} = \widetilde{U} \times V_0 \times V_3$. We invoke a second lemma that applies to the images of $Z_0$ and $\bigcap_{j=1}^{q} Z_j$ under the projection

$$\pi'_{\widetilde{X}} : \widetilde{X} \to \widetilde{U} \times V_0.$$

Now, $\widetilde{W^{\alpha}} = \pi'_{\widetilde{X}}(Z_0)$, and let

$$\widetilde{W^{\boldsymbol{\beta}}}_* = \pi'_{\widetilde{X}}\left(\bigcap_{j=1}^{q} Z_j\right) = \Sigma_{\beta_1} \times \ldots \times \Sigma_{\beta_q} \times V_0, \tag{5.14}$$

where $\Sigma_{\beta_j} \subset U_j^{(1)}$ for each $j$.

**Lemma 5.3.2.** *Let $\widetilde{X} = \widetilde{U} \times V_0 \times V_3$. Suppose $Z_0$ is transverse to $\bigcap_{j=1}^{q} Z_j$ in $\widetilde{X}$, with $\pi'_{\widetilde{X}}|Z_0 \to \widetilde{W^{\alpha}}$ a fibration and $\pi'_{\widetilde{X}}|\bigcap_{j=1}^{q} Z_j \to \widetilde{W^{\boldsymbol{\beta}}}_*$ a diffeomorphism. Then $\widetilde{W^{\boldsymbol{\beta}}}_*$ is transverse to $\widetilde{W^{\alpha}}$ in $\widetilde{U} \times V_0$.*

Proof. Let $w \in \widetilde{W^{\alpha}} \bigcap \widetilde{W^{\boldsymbol{\beta}}}_*$. Then for any $v \in V_3$, $(w, v) \in Z_0$. Since $w \in \widetilde{W^{\boldsymbol{\beta}}}_*$, there exists $w' \in \bigcap_{j=1}^{q} Z_j$ such that $\pi'_{\widetilde{X}}(w') = w$. Then $w' = (w, v')$ for some $v' \in V_3$, which implies that $w' \in Z_0$. Then we have

$$T_{w'} Z_0 + T_{w'}\left(\widetilde{W^{\boldsymbol{\beta}}} \times V_0\right) = T_{w'}\widetilde{X}.$$

Applying $d\pi'_{\widetilde{X}}$, we obtain at the point $w$

$$T_w \widetilde{W^{\alpha}} + T_w \widetilde{W^{\boldsymbol{\beta}}}_* = T_w(\widetilde{U} \times V_0).$$

This proves the lemma. $\square$

Finally, let $\pi_{\widetilde{U}} : \widetilde{U} \times V_0 \to \widetilde{U}$, and we consider the projection of $\widetilde{W^{\boldsymbol{\beta}}}_*$ under $\pi_{\widetilde{U}}$, which is a fibration. Let

$$\widetilde{W^{\boldsymbol{\beta}}}_{\widetilde{U}} = \Sigma'_{\beta_1} \times \ldots \times \Sigma'_{\beta_q} \tag{5.15}$$

denote this projection, and consider the diffeomorphism

$$\pi_{\widetilde{U}}|\widetilde{W^{\alpha}} \to \widetilde{W^{\alpha}}_{\widetilde{U}}.$$

Then we apply Lemma 5.3.2 a second time to conclude that $\Sigma'_{\beta_1} \times \ldots \times \Sigma'_{\beta_q}$ is transverse to $\Sigma'_{\alpha}$ in $\widetilde{U}$. Therefore, their images in $\mathcal{B}$ under the diffeomorphism are transverse, which shows that $\mathcal{B}$ is Whitney stratified by the transverse intersections of the strata corresponding to the external and internal medial axes.

We conclude this section by summarizing the consequences of Theorem 4.3.1 applied to the elements of ${}_{\ell}\mathcal{S}$. First, transversality to the submanifolds representing orbits of simple multigerms (see Sections 5.2.1 and 5.2.4) implies the following:

(1) For $n \leq 6$, every region $\Omega_i \subset \mathbb{R}^{n+1}, i = 1, \ldots, q$, within a generic Blum medial linking structure has a Blum medial axis exhibiting only the generic local normal forms given in Theorem 1.4.4. Furthermore, the linking medial axis is generic.

Second, we summarize the consequences of Theorem 4.3.1 applied to the distinguished submanifolds $W^{(\alpha:\boldsymbol{\beta})}$ in Section 5.2.2 and the stratified spaces in Section 5.2.4:

(2) For $n \leq 6$, a generic Blum medial linking structure exhibits the generic linking configurations given in Theorems 3.4.3, 3.4.4, and 5.5.1.

As explained in Section 5.2.3, there are also consequences for self–linking as a result of transversality to submanifolds which involve generic properties for the geometry of hypersurfaces.

The classification theorems now follow from Theorem 4.3.1 and the work in this section.

## 5.4. Proof of Theorems 3.5.3, 3.5.6, and 3.5.7 in Chapter 3

In this section, we prove three theorems from Section 3.5 in Chapter 3 on the generic stratifications by linking type of the boundaries and medial axes.

First, to prove Theorem 3.5.3, we must show that the stratification $\mathscr{S}^{\mathcal{B}}$ in Definition 3.5.1 of Chapter 3 forms a Whitney stratification of $\mathcal{B}$ in dimensions $n \leq 6$.

Proof of Theorem 3.5.3. Let $\phi \in \mathcal{P} \subset \mathrm{DEmb}(X, \mathbb{R}^{n+1})$, and let $\mathscr{S}^{\mathcal{B}}_{\sigma_0}$ and $\mathscr{S}^{\mathcal{B}}_{\sigma_i}$ denote the Whitney stratifications of $\mathcal{B} = \phi(X)$ corresponding, respectively, to the Maxwell set of $\sigma_0$ and the Maxwell set of $\sigma_i$ for $i = 1, \ldots, q$. Consider the stratum $(\Sigma_\alpha : \Sigma_{\boldsymbol{\beta}})^{\mathcal{B}} \in \mathscr{S}^{\mathcal{B}}$ corresponding to the linking configuration $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$. Then $\Sigma_\alpha$ is a stratum in $\mathscr{S}^{\mathcal{B}}_{\sigma_0}$, while $\Sigma_{\beta_i}$ is a stratum in $\mathscr{S}^{\mathcal{B}}_{\sigma_i}$ for every $i$. Theorem 4.3.1 implies that $\Sigma_\alpha$

and $\Sigma_{\beta_i}$ intersect transversely in $\mathcal{B}$ for all $i$, as explained in Section 5.3. Therefore, since the stratifications $\mathscr{S}^{\mathcal{B}}_{\sigma_0}$ and $\mathscr{S}^{\mathcal{B}}_{\sigma_i}$ are Whitney regular and their strata have transverse intersections in $\mathcal{B}$, a well–known theorem implies that their intersection, $\mathscr{S}^{\mathcal{B}}$, is a Whitney stratification of $\mathcal{B}$. $\square$

Next, we prove Theorem 3.5.6 by showing that in $\mathbb{R}^2$ and $\mathbb{R}^3$, $\mathscr{S}^{M_0}$ is a Whitney stratification of $M_0$.

PROOF OF THEOREM 3.5.6. We must show that $\mathscr{S}^{M_0}$ is a Whitney stratification by showing that all pairs of strata $(\chi_{\mathcal{A}_\alpha} : \chi_{\mathcal{A}_\beta}), (\chi_{\mathcal{A}_\alpha} : \chi_{\mathcal{A}_\beta})'$ with $(\chi_{\mathcal{A}_\alpha} : \chi_{\mathcal{A}_\beta}) \subset \overline{(\chi_{\mathcal{A}_\alpha} : \chi_{\mathcal{A}_\beta})'}$ satisfy Whitney's condition (b) as defined in Definition 1.5.2 in Chapter 1.

First, let $\phi : X \to \mathbb{R}^2$ with $\phi \in \mathcal{P}$. In this case, Whitney's condition (b) holds for all pairs of strata in $\mathscr{S}^{M_0}$ since every stratum (except the smooth one corresponding to the configuration $(A_1^2 : A_1^2, A_1^2)$) is simply a distinguished point on a smooth curve, or a singular point on the Whitney stratification of $M_0$.

Second, let $\phi : X \to \mathbb{R}^3$ with $\phi \in \mathcal{P}$. We begin by pointing out that the strata of $\mathscr{S}^{M_0}$ associated to the configurations $(A_1^2 : A_1^2, A_1^2)$, $(A_1^3 : A_1^2, A_1^2, A_1^2)$, $(A_1^4 : A_1^2, A_1^2, A_1^2, A_1^2)$, $(A_1A_3 : A_1^2, A_1^2)$, and $(A_3 : A_1^2)$ correspond to open subsets in the Whitney stratification of $M_0$. Next, consider the strata $(\chi_{A_1^2} : \chi_{A_1^3}, \chi_{A_1^2})^{M_0}$ and $(\chi_{A_1^2} : \chi_{A_3}, \chi_{A_1^2})^{M_0}$ associated to the configurations $(A_1^2 : A_1^3, A_1^2)$ and $(A_1^2 : A_3, A_1^2)$, respectively, both of which are contained in $\overline{(\chi_{A_1^2} : \chi_{A_1^2}, \chi_{A_1^2})^{M_0}}$. For sequences of points $\{y_i\}$ in $(\chi_{A_1^2} : \chi_{A_1^3}, \chi_{A_1^2})^{M_0}$ (resp., $\{y_i\}$ in $(\chi_{A_1^2} : \chi_{A_3}, \chi_{A_1^2})^{M_0}$) and $\{x_i\}$ in $(\chi_{A_1^2} : \chi_{A_1^2}, \chi_{A_1^2})^{M_0}$ with $\{x_i\}, \{y_i\}$ both converging to a point $y \in (\chi_{A_1^2} : \chi_{A_3}, \chi_{A_1^2})^{M_0}$, condition (b) holds since the limit of the secant lines $\overline{x_i y_i}$ is contained in the tangent space to the smooth stratum of $M_0$.

Likewise, consider the strata of $\mathscr{S}^{M_0}$ associated to the configurations $(A_1^2 : A_1^4, A_1^2)$ and $(A_1^2 : A_1A_3, A_1^2)$. The stratum $(\chi_{A_1^2} : \chi_{A_1^4}, \chi_{A_1^2})^{M_0} \subset \overline{(\chi_{A_1^2} : \chi_{A_1^3}, \chi_{A_1^2})^{M_0}}$, and the stratum $(\chi_{A_1^2} : \chi_{A_1A_3}, \chi_{A_1^2})^{M_0}$ is contained in both $\overline{(\chi_{A_1^2} : \chi_{A_1^3}, \chi_{A_1^2})^{M_0}}$ and $\overline{(\chi_{A_1^2} : \chi_{A_3}, \chi_{A_1^2})^{M_0}}$. The fact that Whitney's condition (b) holds for these pairs of strata is due to the fact that the ordinary stratifications on the individual medial axes are Whitney.

Finally, we consider those strata associated to the 6 remaining configurations in Theorem 3.4.4: $(A_1^2 : A_1^3, A_1^3), (A_1^2 : A_1^3, A_3), (A_1^2 : A_3, A_3), (A_1^3 : A_1^3, A_1^2, A_1^2), (A_1^3 : A_3, A_1^2, A_1^2), (A_3 : A_1^3)$, and $(A_3 : A_3)$. In each case, the $\chi_{\mathcal{A}_\alpha}$ stratum on $M_0$ and the images in $M_0$ of the $\chi_{\mathcal{A}_{\beta_i}}$ strata on $M_i$ are transverse, since Theorem 4.3.1 implies that the corresponding strata on the boundaries, $\Sigma_{\mathcal{A}_\alpha}$ and $\Sigma_{\mathcal{A}_{\beta_i}}$ for every $i$, are transverse in $\mathcal{B}$. Each of these 6 strata in $\mathscr{S}^{M_0}$ is a point occurring at the transverse intersection of two smooth curves. If, in each case, $y$ denotes the point and $\{x_i\}$ is a sequence of points in one of the curves converging to $y$, then the limit of the secant lines $\overline{x_i y}$ is the tangent line to the curve at $y$ because the curve is smooth. We conclude that Whitney's condition (b) holds for all pairs of strata in $\mathscr{S}^{M_0}$. $\square$

Theorem 3.5.7 establishes the generic properties of the stratifications in Theorems 3.5.3 and 3.5.6 for $n \leq 6$, namely:

(1) The number of generically appearing stratum types is finite.

(2) The codimension in $\mathcal{B}$ of the stratum $(\Sigma_\alpha : \Sigma_{\boldsymbol{\beta}})^{\mathcal{B}} \in \mathscr{S}^{\mathcal{B}}$ representing the generic linking configuration $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$ is given by the formula

$$\operatorname{codim}_{\mathcal{B}}((\Sigma_\alpha : \Sigma_{\boldsymbol{\beta}})^{\mathcal{B}}) = \mathcal{R}_e^+\text{-codim}(\mathcal{A}_\alpha) + \sum_{i=1}^{\ell} \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\beta_i}) - (q(\boldsymbol{\ell}) + 1).$$

(3) The codimension in $\mathbb{R}^{n+1}$ of both the stratum $(\chi_\alpha : \chi_{\boldsymbol{\beta}})^{\widetilde{M}} \in \mathscr{S}^{\widetilde{M}}$ and the stratum $(\chi_\alpha : \chi_{\boldsymbol{\beta}})^{M_0} \in \mathscr{S}^{M_0}$ corresponding to $(\mathcal{A}_\alpha : \mathcal{A}_{\beta_1}, \ldots, \mathcal{A}_{\beta_\ell})$ is given by

$$\operatorname{codim}_{\mathbb{R}^{n+1}}(\chi_\alpha : \chi_{\boldsymbol{\beta}}) = \mathcal{R}_e^+\operatorname{codim}(\mathcal{A}_\alpha) + \sum_{i=1}^{\ell} \mathcal{R}_e^+\operatorname{codim}(\mathcal{A}_{\beta_i}) - q(\boldsymbol{\ell}).$$

Proof of Theorem 3.5.7. Let $\phi \in \mathcal{P}$. Property 1 is a consequence of Mather's finite classification in Theorem 1.4.4 of the generic local normal forms of the Blum medial axis in dimensions $n \leq 6$. To prove property 2, let $W^{(\alpha:\boldsymbol{\beta})}$ be the submanifold of ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$ associated to the configuration $(\mathcal{A}_\alpha : \mathcal{A}_{\boldsymbol{\beta}})$. Let $\operatorname{codim}_{{}_{\boldsymbol{\ell}}E^{(k)}}(W^{(\alpha:\boldsymbol{\beta})})$ denote the codimension of $W^{(\alpha:\boldsymbol{\beta})}$ in ${}_{\boldsymbol{\ell}}E^{(k)}(X, \mathbb{R}^2)$. For $\phi \in \mathcal{P}$, Theorem 4.3.1 implies that $\operatorname{codim}_{{}_{\boldsymbol{\ell}}E^{(k)}}(W^{(\alpha:\boldsymbol{\beta})})$ equals the codimension of ${}_{\boldsymbol{\ell}}j_1^k \rho_\phi^{-1}(W^{(\alpha:\boldsymbol{\beta})})$ in the space

$X^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$, and thus also in the space $\mathcal{B}^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}$. Now

$$\operatorname{codim}_{\boldsymbol{\ell}E^{(k)}}(W^{(\alpha:\boldsymbol{\beta})}) = \operatorname{codim}_{q(\boldsymbol{\ell})J^k(X,\mathbb{R})}(W^{\alpha}) + \sum_{i=1}^{q} \operatorname{codim}_{\ell_i J^k(X_i,\mathbb{R})}(W^{\beta_i}), \tag{5.16}$$

and recall equation (1.17) in Section 1.6 in Chapter 1, which states that the codimension in ${}_rJ^k(X,\mathbb{R})$ of $W^{\eta}$ may be written in terms of $\mathcal{R}_e^+ - \operatorname{codim}(\mathcal{A}_\eta)$:

$$\operatorname{codim}(W^{\eta}) \text{ in } {}_rJ^k(X,\mathbb{R}) = \mathcal{R}_e^+\text{–}\operatorname{codim}(\mathcal{A}_\eta) + nr, \tag{5.17}$$

where the multi-germ $\mathcal{A}_\eta$ is an $r$–tuple. Using equations (5.16) and (5.17), we find that $\dim(\mathcal{B}^{(\boldsymbol{\ell})} \times (\mathbb{R}^{n+1})^{(q(\boldsymbol{\ell})+1)}) - \operatorname{codim}_{\boldsymbol{\ell}E^{(k)}}(W^{(\alpha:\boldsymbol{\beta})})$ is given by

$$\begin{aligned}
&\sum_{i=1}^{q} n\ell_i + (n+1)(q(\boldsymbol{\ell})+1) - nq(\boldsymbol{\ell}) - \sum_{i=1}^{q} n\ell_i - \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\boldsymbol{\alpha}}) - \\
&\qquad\qquad\qquad\qquad\qquad\qquad \sum_{i=1}^{q} \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\boldsymbol{\beta}_i}) \\
&= (n+1)(q(\boldsymbol{\ell})+1) - nq(\boldsymbol{\ell}) - \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\boldsymbol{\alpha}}) - \sum_{i=1}^{q} \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\boldsymbol{\beta}_i}) \\
&= n - \left( \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\boldsymbol{\alpha}}) + \sum_{i=1}^{q} \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\boldsymbol{\beta}_i}) - (q(\boldsymbol{\ell})+1) \right).
\end{aligned}$$

Thus, the codimension in $\mathcal{B}$ of its projection, denoted $\operatorname{codim}_{\mathcal{B}}((\Sigma_\alpha : \Sigma_{\boldsymbol{\beta}})^{\mathcal{B}})$, is given by

$$\operatorname{codim}_{\mathcal{B}}((\Sigma_\alpha : \Sigma_{\boldsymbol{\beta}})^{\mathcal{B}}) = \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\boldsymbol{\alpha}}) + \sum_{i=1}^{q} \mathcal{R}_e^+\text{-codim}(\mathcal{A}_{\boldsymbol{\beta}_i}) - (q(\boldsymbol{\ell})+1). \tag{5.18}$$

This proves property 2, and property 3 follows immediately. □

## 5.5. Classification of generic linking for $\mathbb{R}^4$ through $\mathbb{R}^7$

In this section, we present the classification of generic medial linking configurations of codimension $\leq n+1$ in $\mathbb{R}^{n+1}$ for $3 \leq n \leq 6$. The complete classification for dimensions $3 \leq n \leq 6$ includes all configurations of codimension $\leq n+1$ in $\mathbb{R}^{n+1}$, but we only present the new configurations of codimension $n+1$ in each successive dimension. The classification results are given in Tables 5.1 — 5.4. In Table 5.4, we use the notation

"$-$" to indicate all $A_1^2$ linking within a linking configuration. We label with a "$*$" those self–linking configurations that are not shown to be generic in this dissertation.

**Theorem 5.5.1 (Classification of generic linking in $\mathbb{R}^{n+1}$ for $3 \leq n \leq 6$).** *For a residual set of embeddings of smooth, disjoint, compact, connected hypersurfaces whose interiors bound disjoint regions in $\mathbb{R}^{n+1}$ for $3 \leq n \leq 6$, the generic linking types include all generic linking types in $\mathbb{R}^n$, as well as the generic linking types of codimension $n+1$ in $\mathbb{R}^{n+1}$ which are given in Tables 5.1–5.4. Within each of the four tables, the configurations above the double horizontal lines can occur for linking between distinct regions, while all configurations in the tables can occur for self-linking.*

TABLE 5.1. Generic linking type of codimension 4 in $\mathbb{R}^4$.

| Linking type | Configuration |
|---|---|
| I | $(A_1^2 : A_1^2, A_5)$<br>$(A_1^2 : A_1^2, A_1^5)$<br>$(A_1^2 : A_1^2, A_1^2A_3)$<br>$(A_1^2 : A_1^3, A_1^4)$<br>$(A_1^2 : A_1^3, A_1A_3)$<br>$(A_1^2 : A_3, A_1^4)$<br>$(A_1^2 : A_3, A_1A_3)$ |
| II | $(A_1^5 : A_1^2, A_1^2, A_1^2, A_1^2, A_1^2)$<br>$(A_1^2A_3 : A_1^2, A_1^2, A_1^2)$ |
| III | $(A_1^3 : A_1^2, A_1^3, A_1^3)$<br>$(A_1^3 : A_1^2, A_1^3, A_3)$<br>$(A_1^3 : A_1^2, A_3, A_3)$<br>$(A_1^3 : A_1^2, A_1^2, A_1^4)$<br>$(A_1^3 : A_1^2, A_1^2, A_1A_3)$<br>$(A_1^4 : A_1^2, A_1^2, A_1^2, A_1^3)$<br>$(A_1^4 : A_1^2, A_1^2, A_1^2, A_3)$<br>$(A_1A_3 : A_1^2, A_1^3)$<br>$(A_1A_3 : A_1^2, A_3)$ |
| II | $(A_5 : A_1^2)$ |
| III | $(A_3 : A_1^4)$<br>$(A_3 : A_1A_3)$ |

TABLE 5.2. Generic linking type of codimension 5 in $\mathbb{R}^5$.

| Linking type | Configuration | |
|---|---|---|
| I | $(A_1^2 : A_3, A_5)$ | $(A_1^2 : A_3, A_1^2A_3)$ |
| | $(A_1^2 : A_3, A_1^5)$ | $(A_1^2 : A_1^2, A_3^2)$ |
| | $(A_1^2 : A_1^3, A_5)$ | $(A_1^2 : A_1^2, A_1^6)$ |
| | $(A_1^2 : A_1^3, A_1^5)$ | $(A_1^2 : A_1^2, A_1A_5)$ |
| | $(A_1^2 : A_1^3, A_1^2A_3)$ | $(A_1^2 : A_1^2, A_1^3A_3)$ |
| | $(A_1^2 : A_1^4, A_1^4)$ | $(A_1^2 : A_1A_3, A_1A_3)$ |
| | $(A_1^2 : A_1A_3, A_1^4)$ | |
| II | $(A_1^6 : A_1^2, A_1^2, A_1^2, A_1^2, A_1^2, A_1^2)$ | $(A_1A_5 : A_1^2, A_1^2)$ |
| | $(A_3^2 : A_1^2, A_1^2)$ | $(A_1^3A_3 : A_1^2, A_1^2, A_1^2, A_1^2)$ |
| III | $(A_1^3 : A_1^3, A_1^3, A_1^3)$ | $(A_1^3 : A_3, A_3, A_1^3)$ |
| | $(A_1^3 : A_3, A_1^3, A_1^3)$ | $(A_1^3 : A_3, A_3, A_3)$ |
| | $(A_1^3 : A_1^2, A_1^3, A_1^4)$ | $(A_1^3 : A_1^2, A_3, A_1^4)$ |
| | $(A_1^3 : A_1^2, A_1^3, A_1A_3)$ | $(A_1^3 : A_1^2, A_3, A_1A_3)$ |
| | $(A_1^3 : A_1^2, A_1^2, A_1^5)$ | $(A_1^3 : A_1^2, A_1^2, A_5)$ |
| | $(A_1^3 : A_1^2, A_1^2, A_1^2A_3)$ | $(A_1^4 : A_1^3, A_1^3, A_1^2, A_1^2)$ |
| | $(A_1^4 : A_1^3, A_3, A_1^2, A_1^2)$ | $(A_1^4 : A_3, A_3, A_1^2, A_1^2)$ |
| | $(A_1^4 : A_1^4, A_1^2, A_1^2, A_1^2)$ | $(A_1^4 : A_1A_3, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1A_3 : A_1^3, A_1^3)$ | $(A_1A_3 : A_1^3, A_3)$ |
| | $(A_1A_3 : A_3, A_3)$ | $(A_1A_3 : A_1^4, A_1^2)$ |
| | $(A_1A_3 : A_1A_3, A_1^2)$ | $(A_1^5 : A_1^3, A_1^2, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1^5 : A_3, A_1^2, A_1^2, A_1^2, A_1^2)$ | $(A_1^2A_3 : A_1^3, A_1^2, A_1^2)$ |
| | $(A_1^2A_3 : A_3, A_1^2, A_1^2)$ | |
| III | $(A_3 : A_1^2A_3)$ | $(A_3 : A_5)*$ |
| | $(A_5 : A_1^3)$ | $(A_5 : A_3)*$ |

TABLE 5.3. Generic linking type of codimension 6 in $\mathbb{R}^6$.

| Linking type | Configuration | |
|---|---|---|
| I | $(A_1^2 : A_1^4, A_1^5)$ | $(A_1^2 : A_1^4, A_5)$ |
| | $(A_1^2 : A_1^4, A_1^2A_3)$ | $(A_1^2 : A_1A_3, A_1^2A_3)$ |
| | $(A_1^2 : A_1A_3, A_1^5)$ | $(A_1^2 : A_1A_3, A_5)$ |
| | $(A_1^2 : A_1^3, A_1^6)$ | $(A_1^2 : A_1^3, A_1A_5)$ |
| | $(A_1^2 : A_1^3, A_3^2)$ | $(A_1^2 : A_1^3, A_1^3A_3)$ |
| | $(A_1^2 : A_3, A_1^6)$ | $(A_1^2 : A_3, A_1A_5)$ |
| | $(A_1^2 : A_3, A_3^2)$ | $(A_1^2 : A_3, A_1^3A_3)$ |
| | $(A_1^2 : A_1^2, A_1^7)$ | $(A_1^2 : A_1^2, A_1A_3^2)$ |
| | $(A_1^2 : A_1^2, A_1^2A_5)$ | $(A_1^2 : A_1^2, A_1^4A_3)$ |
| | $(A_1^2 : A_7, A_1^2)$ | |
| II | $(A_1^7 : A_1^2, A_1^2, A_1^2, A_1^2, A_1^2, A_1^2, A_1^2)$ | $(A_1A_3^2 : A_1^2, A_1^2, A_1^2)$ |
| | $(A_1^2A_5 : A_1^2, A_1^2, A_1^2)$ | $(A_1^4A_3 : A_1^2, A_1^2, A_1^2, A_1^2, A_1^2)$ |
| III | $(A_1^3 : A_1^4, A_1^3, A_1^3)$ | $(A_1^3 : A_1^4, A_3, A_1^3)$ |
| | $(A_1^3 : A_1^4, A_3, A_3)$ | $(A_1^3 : A_1A_3, A_3, A_3)$ |
| | $(A_1^3 : A_1A_3, A_1^3, A_1^3)$ | $(A_1^3 : A_1A_3, A_3, A_1^3)$ |
| | $(A_1^3 : A_1^2, A_1^3, A_1^5)$ | $(A_1^3 : A_1^2, A_1^3, A_1^2A_3)$ |
| | $(A_1^3 : A_1^2, A_1^3, A_5)$ | $(A_1^3 : A_1^2, A_1^4, A_1^4)$ |
| | $(A_1^3 : A_1^2, A_1^4, A_1A_3)$ | $(A_1^3 : A_1^2, A_1A_3, A_1A_3)$ |
| | $(A_1^3 : A_1^2, A_3, A_1^5)$ | $(A_1^3 : A_1^2, A_3, A_1^2A_3)$ |
| | $(A_1^3 : A_1^2, A_3, A_5)$ | $(A_1^3 : A_1^2, A_1^2, A_1^6)$ |
| | $(A_1^3 : A_1^2, A_1^2, A_3^2)$ | $(A_1^3 : A_1^2, A_1^2, A_1A_5)$ |
| | $(A_1^3 : A_1^2, A_1^2, A_1^3A_3)$ | $(A_1^4 : A_1^4, A_1^3, A_1^2, A_1^2)$ |
| | $(A_1^4 : A_1^3, A_1A_3, A_1^2, A_1^2)$ | $(A_1^4 : A_3, A_1^4, A_1^2, A_1^2)$ |
| | $(A_1^4 : A_1A_3, A_3, A_1^2, A_1^2)$ | $(A_1^4 : A_1^5, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1^4 : A_1^2A_3, A_1^2, A_1^2, A_1^2)$ | $(A_1^4 : A_5, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1A_3 : A_1^4, A_1^3)$ | $(A_1A_3 : A_1^3, A_1A_3)$ |
| | $(A_1A_3 : A_3, A_1^4)$ | $(A_1A_3 : A_1A_3, A_3)$ |
| | $(A_1A_3 : A_1^5, A_1^2)$ | $(A_1A_3 : A_1^2A_3, A_1^2)$ |
| | $(A_1A_3 : A_5, A_1^2)$ | $(A_1^5 : A_1^3, A_1^3, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1^5 : A_3, A_1^3, A_1^2, A_1^2, A_1^2)$ | $(A_1^5 : A_3, A_3, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1^5 : A_1^4, A_1^2, A_1^2, A_1^2, A_1^2)$ | $(A_1^5 : A_1A_3, A_1^2, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1^2A_3 : A_1^3, A_1^3, A_1^2)$ | $(A_1^2A_3 : A_3, A_1^3, A_1^2)$ |
| | $(A_1^2A_3 : A_3, A_3, A_1^2)$ | $(A_1^2A_3 : A_1^4, A_1^2, A_1^2)$ |
| | $(A_1^2A_3 : A_1A_3, A_1^2, A_1^2)$ | $(A_1^6 : A_1^3, A_1^2, A_1^2, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1^6 : A_3, A_1^2, A_1^2, A_1^2, A_1^2, A_1^2)$ | $(A_1A_5 : A_1^3, A_1^2)$ |
| | $(A_1A_5 : A_3, A_1^2)$ | $(A_3^2 : A_1^3, A_1^2)$ |
| | $(A_3^2 : A_3, A_1^2)$ | $(A_1^3A_3 : A_1^3, A_1^2, A_1^2, A_1^2)$ |
| | $(A_1^3A_3 : A_3, A_1^2, A_1^2, A_1^2)$ | |
| II | $(A_7 : A_1^2)$ | |
| III | $(A_3 : A_1^3A_3)$ | $(A_3 : A_3^2)$ |
| | $(A_3 : A_1A_5)$ | $(A_5 : A_1^4)$ |
| | $(A_5 : A_1A_3)$ | |

Table 5.4. Generic linking type of codimension 7 in $\mathbb{R}^7$.

| Linking type | Configuration | | |
|---|---|---|---|
| I | $(A_1^2 : A_1^5, A_1^5)$ | $(A_1^2 : A_1^5, A_1^2A_3)$ | $(A_1^2 : A_1^5, A_5)$ |
| | $(A_1^2 : A_1^2A_3, A_1^2A_3)$ | $(A_1^2 : A_1^2A_3, A_5)$ | $(A_1^2 : A_5, A_5)$ |
| | $(A_1^2 : A_1^6, A_1^4)$ | $(A_1^2 : A_1^6, A_1A_3)$ | $(A_1^2 : A_1A_5, A_1^4)$ |
| | $(A_1^2 : A_1A_5, A_1A_3)$ | $(A_1^2 : A_3^2, A_1^4)$ | $(A_1^2 : A_3^2, A_1A_3)$ |
| | $(A_1^2 : A_1^3A_3, A_1^4)$ | $(A_1^2 : A_1^3A_3, A_1A_3)$ | $(A_1^2 : A_1^7, A_1^3)$ |
| | $(A_1^2 : A_1^7, A_3)$ | $(A_1^2 : A_1A_3^2, A_1^3)$ | $(A_1^2 : A_1A_3^2, A_3)$ |
| | $(A_1^2 : A_1^2A_5, A_1^3)$ | $(A_1^2 : A_1^2A_5, A_3)$ | $(A_1^2 : A_1^4A_3, A_1^3)$ |
| | $(A_1^2 : A_1^4A_3, A_3)$ | $(A_1^2 : A_7, A_1^3)$ | $(A_1^2 : A_7, A_3)$ |
| | $(A_1^2 : A_1^8, -)$ | $(A_1^2 : A_1^3A_5, -)$ | $(A_1^2 : A_1^2A_3^2, -)$ |
| | $(A_1^2 : A_1^5A_3, -)$ | $(A_1^2 : A_1A_7, -)$ | $(A_1^2 : A_3A_5, -)$ |
| II | $(A_1^8 : -)$ | $(A_1^3A_5 : A_1^3, -)$ | $(A_1^2A_3^2 : -)$ |
| | $(A_1^5A_3 : -)$ | $(A_1A_7 : -)$ | $(A_3A_5 : -)$ |
| III | $(A_1^3 : A_1^5, A_1^3, A_1^3)$ | $(A_1^3 : A_1^5, A_3, A_1^3)$ | $(A_1^3 : A_1^5, A_3, A_3)$ |
| | $(A_1^3 : A_1^2A_3, A_1^3, A_1^3)$ | $(A_1^3 : A_1^2A_3, A_1^3, A_3)$ | $(A_1^3 : A_1^2A_3, A_3, A_3)$ |
| | $(A_1^3 : A_5, A_1^3, A_1^3)$ | $(A_1^3 : A_5, A_3, A_1^3)$ | $(A_1^3 : A_5, A_3, A_3)$ |
| | $(A_1^3 : A_1^6, A_1^3, -)$ | $(A_1^3 : A_1^6, A_3, -)$ | $(A_1^3 : A_1A_5, A_1^3, -)$ |
| | $(A_1^3 : A_1A_5, A_3, -)$ | $(A_1^3 : A_3^2, A_1^3, -)$ | $(A_1^3 : A_3^2, A_3, -)$ |
| | $(A_1^3 : A_1^3A_3, A_1^3, -)$ | $(A_1^3 : A_1^3A_3, A_3, -)$ | $(A_1^3 : A_1^4, A_1^4, A_1^3)$ |
| | $(A_1^3 : A_1^4, A_1^4, A_3)$ | $(A_1^3 : A_1^4, A_1A_3, A_1^3)$ | $(A_1^3 : A_1^4, A_1A_3, A_3)$ |
| | $(A_1^3 : A_1A_3, A_1A_3, A_1^3)$ | $(A_1^3 : A_1A_3, A_1A_3, A_3)$ | $(A_1^3 : A_1^5, A_1^4, -)$ |
| | $(A_1^3 : A_1^5, A_1A_3, -)$ | $(A_1^3 : A_1^2A_3, A_1^4, -)$ | $(A_1^3 : A_1^2A_3, A_1A_3, -)$ |
| | $(A_1^3 : A_5, A_1^4, -)$ | $(A_1^3 : A_5, A_1A_3, -)$ | $(A_1^3 : A_1^7, -)$ |
| | $(A_1^3 : A_1A_3^2, -)$ | $(A_1^3 : A_1^2A_5, -)$ | $(A_1^3 : A_1^4A_3, -)$ |
| | $(A_1^4 : A_1^3, A_1^3, A_1^3, A_1^3)$ | $(A_1^4 : A_1^3, A_1^3, A_1^3, A_3)$ | $(A_1^4 : A_1^3, A_1^3, A_3, A_3)$ |
| | $(A_1^4 : A_1^3, A_3, A_3, A_3)$ | $(A_1^4 : A_3, A_3, A_3, A_3)$ | $(A_1^4 : A_1A_3, A_1^3, A_1^3, -)$ |
| | $(A_1^4 : A_1A_3, A_1^3, A_3, -)$ | $(A_1^4 : A_1A_3, A_3, A_3, -)$ | $(A_1^4 : A_1^4, A_1^3, A_1^3, -)$ |
| | $(A_1^4 : A_1^4, A_1^3, A_3, -)$ | $(A_1^4 : A_1^4, A_3, A_3, -)$ | $(A_1^4 : A_1A_3, A_1A_3, -)$ |
| | $(A_1^4 : A_1^4, A_1^4, -)$ | $(A_1^4 : A_1^4, A_1A_3, -)$ | $(A_1^4 : A_1^5, A_1^3, -)$ |
| | $(A_1^4 : A_1^5, A_3, -)$ | $(A_1^4 : A_1^2A_3, A_1^3, -)$ | $(A_1^4 : A_1^2A_3, A_3, -)$ |
| | $(A_1^4 : A_5, A_1^3, -)$ | $(A_1^4 : A_5, A_3, -)$ | $(A_1^4 : A_1^6, -)$ |
| | $(A_1^4 : A_1^3A_3, -)$ | $(A_1^4 : A_1A_5, -)$ | $(A_1^4 : A_3^2, -)$ |
| | $(A_1A_3 : A_1^5, A_1^3)$ | $(A_1A_3 : A_1^5, A_3)$ | $(A_1A_3 : A_1^2A_3, A_1^3)$ |
| | $(A_1A_3 : A_1^2A_3, A_3)$ | $(A_1A_3 : A_5, A_1^3)$ | $(A_1A_3 : A_5, A_3)$ |
| | $(A_1A_3 : A_1^6, -)$ | $(A_1A_3 : A_1^3A_3, -)$ | $(A_1A_3 : A_1A_5, -)$ |
| | $(A_1A_3 : A_3^2, -)$ | $(A_1A_3 : A_1^4, A_1^4)$ | $(A_1A_3 : A_1^4, A_1A_3)$ |
| | $(A_1A_3 : A_1A_3, A_1A_3)$ | $(A_1^5 : A_1^3, A_1^3, A_1^3, -)$ | $(A_1^5 : A_1^3, A_1^3, A_3, -)$ |
| | $(A_1^5 : A_1^3, A_3, A_3, -)$ | $(A_1^5 : A_3, A_3, A_3, -)$ | $(A_1^5 : A_1^4, A_1^3, -)$ |
| | $(A_1^5 : A_1^4, A_3, -)$ | $(A_1^5 : A_1A_3, A_1^3, -)$ | $(A_1^5 : A_1A_3, A_3, -)$ |
| | $(A_1^5 : A_1^5, -)$ | $(A_1^5 : A_1^2A_3, -)$ | $(A_1^5 : A_5, -)$ |
| | $(A_1^2A_3 : A_1^3, A_1^3, A_1^3)$ | $(A_1^2A_3 : A_1^3, A_1^3, A_3)$ | $(A_1^2A_3 : A_1^3, A_3, A_3)$ |
| | $(A_1^2A_3 : A_3, A_3, A_3)$ | $(A_1^2A_3 : A_1^4, A_1^3, -)$ | $(A_1^2A_3 : A_1^4, A_3, -)$ |
| | $(A_1^2A_3 : A_1A_3, A_1^3, -)$ | $(A_1^2A_3 : A_1A_3, A_3, -)$ | $(A_1^2A_3 : A_1^5, -)$ |
| | $(A_1^2A_3 : A_1^2A_3, -)$ | $(A_1^2A_3 : A_5, -)$ | $(A_1^6 : A_1^3, A_1^3, -)$ |

Generic linking type of codimension 7 in $\mathbb{R}^7$, continued.

| Linking type | Configuration | | |
|---|---|---|---|
| | $(A_1^6 : A_1^3, A_3, -)$ | $(A_1^6 : A_3, A_3, -)$ | $(A_1^6 : A_1^4, -)$ |
| | $(A_1^6 : A_1A_3, -)$ | $(A_1A_5 : A_1^3, A_1^3)$ | $(A_1A_5 : A_1^3, A_3)$ |
| | $(A_1A_5 : A_3, A_3)$ | $(A_1A_5 : A_1^4, -)$ | $(A_1A_5 : A_1A_3, -)$ |
| | $(A_3^2 : A_1^3, A_1^3)$ | $(A_3^2 : A_1^3, A_3)$ | $(A_3^2 : A_3, A_3)$ |
| | $(A_3^2 : A_1^4, -)$ | $(A_3^2 : A_1A_3, -)$ | $(A_1^3A_3 : A_1^3, A_1^3, -)$ |
| | $(A_1^3A_3 : A_1^3, A_3, -)$ | $(A_1^3A_3 : A_3, A_3, -)$ | $(A_1^3A_3 : A_1^4, -)$ |
| | $(A_1^3A_3 : A_1A_3, -)$ | $(A_1^7 : A_1^3, -)$ | $(A_1^7 : A_3, -)$ |
| | $(A_1A_3^2 : A_1^3, -)$ | $(A_1A_3^2 : A_3, -)$ | $(A_1^2A_5 : A_1^3, -)$ |
| | $(A_1^2A_5 : A_3, -)$ | $(A_1^4A_3 : A_1^3, -)$ | $(A_1^4A_3 : A_3, -)$ |
| III | $(A_3 : A_1^7)$ | $(A_3 : A_1^4A_3)$ | $(A_3 : A_1A_3^2)$ |
| | $(A_3 : A_1^2A_5)$ | $(A_3 : A_7)*$ | $(A_5 : A_1^5)$ |
| | $(A_5 : A_1^2A_3)$ | $(A_5 : A_5)*$ | $(A_7 : A_1^3)$ |
| | $(A_7 : A_3)*$ | | |

CHAPTER 6

# Applications of the Blum Medial Linking Structure

## 6.1. Introduction

Now that we have established the generic properties of the Blum medial linking structure, we shift our focus to applications of the linking structure in multi–region shape analysis. We begin in Section 6.2 by introducing a linking flow on each region's medial axis and deducing its properties. In Section 6.3, we show how the medial geometry in the complement can be computed directly from the medial geometry of the collection of internal medial axes. Then in Section 6.4, we show how to compute integrals over regions in the complement as a sum of integrals over the individual medial axes.

In the second half of the chapter, we begin to address the motivating questions from medical image analysis given in Section 2.6 in Chapter 2. In Section 6.5, we introduce invariants involving aspects of the positional geometry of a collection of regions. We use these invariants, which may be computed directly from the linking structure, to construct a "tiered graph" that reflects the closeness and significance of the regions relative to one another. In Section 6.6, we conclude with a brief discussion of future research directions involving the linking structure.

## 6.2. Construction of the linking flow

Throughout this chapter, we assume that $\phi : X \to \mathbb{R}^{n+1}$ is a generic embedding in $\mathrm{DEmb}(X, \mathbb{R}^{n+1})$ defining a collection of disjoint regions $\{\Omega_i\}_{i=1}^{q}$ each with smooth boundary $\partial\Omega_i = \mathcal{B}_i$, Blum medial axis $M_i$, and radial vector field $U_i$.

In this section, we begin by defining a piecewise smooth local *linking flow* on each $M_i \setminus M_i^{\infty}$, which extends to a piecewise smooth global linking flow on $\widetilde{M}_i \setminus \widetilde{M}_i^{\infty}$. For $i = 1, \ldots, q$, let $U_i = r_i\boldsymbol{u}_i$ and $L_i = \ell_i\boldsymbol{u}_i$ be the radial and linking vector fields on

$M_i \setminus M_i^\infty$. Let $V$ be an open neighborhood of a point $x_0 \in M_i \setminus M_i^\infty$ with a smooth value of the vector field $L_i$ defined on $V$, with $V$ within a local manifold component of $x_0$ if $(x_0, L_i)$ belongs to a singular stratum of $\mathscr{S}^{\widetilde{M}_i}$.

**Definition 6.2.1.** *For a smooth choice of $L_i$ on a neighborhood $V$ of $x_0 \in M_i \setminus M_i^\infty$, the local* ***linking flow on*** $M_i \setminus M_i^\infty$ *is the map*

$$\Phi_i : V \times [0,1] \to \mathbb{R}^{n+1}, \tag{6.1}$$

$$(x,t) \mapsto x + \chi_i(x,t)\boldsymbol{u}_i(x),$$

*where*

$$\chi_i(x,t) = \begin{cases} 2tr_i(x) & 0 \le t \le \dfrac{1}{2} \\ 2(1-t)r_i(x) + (2t-1)\ell_i(x) & \dfrac{1}{2} < t \le 1 \end{cases}.$$

Observe that for $0 \le t \le 1/2$, the flow is a scaled version of the local radial flow, and that the $t = \dfrac{1}{2}$ level surface is the boundary $\mathcal{B}_i$ of $\Omega_i$. Damon established the local nonsingularity of the radial flow in [**14**] (see Section 2.3 in Chapter 2). In the following proposition, we prove that the second half of the linking flow is locally nonsingular at points $x_0 \in M_i \setminus M_i^\infty$ with $(x_0, L_i)$ in the smooth stratum of $\mathscr{S}^{\widetilde{M}_i}$.

**Proposition 6.2.2.** *Let $L_i = \ell_i \boldsymbol{u}_i$ be a smooth value of the linking vector field in a neighborhood $V$ of a point $x_0 \in M_i \setminus M_i^\infty$ with $(x_0, L_i)$ belonging to the smooth stratum of $\mathscr{S}^{\widetilde{M}_i}$. Assume that, on this neighborhood, the radial shape operator $(S_i)_{rad}$ on $M_i$ satisfies the following* radial linking condition*:*

$$\ell_i < min\left\{\frac{1}{\kappa_{r_j}}\right\} \text{ for all positive principal radial curvatures } \kappa_{r_j} \text{ of } (S_i)_{rad}. \tag{6.2}$$

*Then:*

*(1) The restriction $\Phi_i : V \times (1/2, 1] \to \mathbb{R}^{n+1}$ is a local diffeomorphism at $(x_0, t)$.*

*(2) $\Phi_i(\cdot, t) : V \to \mathbb{R}^{n+1}$ is a local embedding at $x_0$ for $1/2 < t \le 1$.*

*(3) $\Phi_i(, t)(V)$ is transverse to the linking line spanned by $L_i$ for $1/2 < t \le 1$.*

Proof. The proof is very similar to the proof of Proposition 4.1 in [**14**].

Let $V$ be a neighborhood of $x_0 \in (M_i \setminus M_i^{\infty})_{\text{reg}}$ with $L_i(x_0)$ a choice of linking vector at $x_0$ and with a smooth extension of $L_i$ defined on $V$. As in Proposition 4.1 in [**14**], choose a basis $\{v_1, \ldots, v_n\}$ for $T_{x_0}M_i$ and write

$$\frac{\partial \boldsymbol{u}}{\partial v_k} = a_k \boldsymbol{u}_i - \sum_{j=1}^{n} s_{jk} v_j, \tag{6.3}$$

where $\boldsymbol{v}$ denotes the column vector with $k$-th entry $v_k$, $(A_i)_{\boldsymbol{v}}$ is a matrix with $k$-th entry $a_k$, and $(S_i)_{\boldsymbol{v}}$ is a matrix representation for $(S_i)_{\text{rad}}$ with $kj$-th entry $s_{kj}$. Then

$$\begin{aligned}
\frac{\partial \Phi_i}{\partial v_k} =& v_k + 2(1-t)\left[\frac{\partial r}{\partial v_k} \cdot \boldsymbol{u}_i + r_i \cdot \frac{\partial \boldsymbol{u}_i}{\partial v_k}\right] + (2t-1)\left[\frac{\partial \ell_i}{\partial v_k} \cdot \boldsymbol{u}_i + \ell_i \cdot \frac{\partial \boldsymbol{u}_i}{\partial v_k}\right] \\
=& v_k + 2(1-t)\left[dr_i(v_k) \cdot \boldsymbol{u}_i + r_i a_k \boldsymbol{u}_i - \sum_{j=1}^{n} r_i s_{jk} v_j\right] + \\
& \qquad (2t-1)\left[d\ell_i(v_k) \cdot \boldsymbol{u}_i + \ell_i a_k \boldsymbol{u}_i - \sum_{j=1}^{n} \ell_i s_{jk} v_j\right] \\
=& \{2(1-t)[dr_i(v_k) + r_i a_k] + (2t-1)[d\ell_i(v_k) + \ell_i a_k]\} \cdot \boldsymbol{u}_i + \\
& \qquad \sum_{j=1}^{n} \{\delta_{kj} - [2(1-t)r_i + (2t-1)\ell_i] s_{jk}\} \cdot v_j.
\end{aligned}$$

In vector notation,

$$\begin{aligned}
\frac{\partial \Phi_i}{\partial v} =& \{2(1-t)[dr_i(\mathbf{v}) + r_i \cdot (A_i)_{\mathbf{v}}] + (2t-1)[d\ell_i(\mathbf{v}) + \ell_i \cdot (A_i)_{\mathbf{v}}]\} \cdot \boldsymbol{u}_i \\
& + \{I - [2(1-t)r_i + (2t-1)\ell_i] \cdot (S_i)_{\mathbf{v}}\}^T \cdot \mathbf{v},
\end{aligned} \tag{6.4}$$

where $I$ is the $n \times n$ identity matrix. Also, note that

$$\frac{\partial \Phi_i}{\partial t} = 2(\ell_i - r_i)(x)\boldsymbol{u}_i(x). \tag{6.5}$$

Next, the transpose matrix of the Jacobian of $\Phi_i$ at $(x_0, t)$ with respect to the bases $\left\{\frac{\partial}{\partial t}, v_1, \ldots, v_n\right\}$ and $\{\boldsymbol{u}_i, v_1, \ldots, v_n\}$ for the source $V \times \mathbb{R}$ at $(x_0, t)$ and target $\mathbb{R}^{n+1}$, respectively, is given by

$$\left(\begin{array}{c|c} 2(\ell_i - r_i) & 0 \\ \hline 2(1-t)[dr_i(\mathbf{v}) + r_i \cdot (A_i)_{\mathbf{v}}]+ & (I - [2(1-t)r_i + (2t-1)\ell_i](S_i)_{\mathbf{v}})^T \\ (2t-1)[d\ell_i(\mathbf{v}) + \ell_i \cdot (A_i)_{\mathbf{v}}] & \end{array}\right),$$

where $\ell_i - r_i$ is nonzero. Also, the transpose of the Jacobian matrix of the flow $\Phi_i(\cdot, t)$, written with respect to the basis $\{v_1, \ldots, v_n\}$ for $T_{x_0}M$ and $\{\boldsymbol{u}_i, v_1, \ldots, v_n\}$ for $\mathbb{R}^{n+1}$, is given by

$$\left(\begin{array}{c|c} 2(1-t)[dr_i(\mathbf{v}) + r_i \cdot (A_i)_{\mathbf{v}}]+ & (I - [2(1-t)r_i + (2t-1)\ell_i](S_i)_{\mathbf{v}})^T \\ (2t-1)[d\ell_i(\mathbf{v}) + \ell_i \cdot (A_i)_{\mathbf{v}}] & \end{array}\right).$$

By the Immersion Theorem, it suffices to show that this matrix has full rank in order for $\Phi_i(\cdot, t)$ to be a local embedding. Now, the matrix

$$I - \{2(1-t)r_i + (2t-1)\ell_i\}(S_i)_{\mathbf{v}} = -\{2(1-t)r_i + (2t-1)\ell_i\}\left[(S_i)_{\mathbf{v}} - \frac{1}{2(1-t)r_i + (2t-1)\ell_i} I\right]$$

will be non-singular if and only if

$$\frac{1}{2(1-t)r_i + (2t-1)\ell_i} \tag{6.6}$$

is not an eigenvalue of $(S_i)_{\mathbf{v}}$ for $1/2 < t \leq 1$. Since at $t = 1/2$, the value of (6.6) is $\dfrac{1}{r_i}$, and at $t = 1$, the value is $\dfrac{1}{\ell_i}$, it follows that all positive eigenvalues of $(S_i)_{\mathbf{v}}$ must not lie in the interval $\left[\dfrac{1}{\ell_i}, \dfrac{1}{r_i}\right)$. But the linking condition (6.2) ensures that all positive eigenvalues $\kappa_{r_j}$ of the radial shape operator are greater than $\dfrac{1}{\ell_i}$. Therefore, we conclude that the two transpose Jacobian matrices above have full rank, which proves statements 1 and 2.

Finally, as in [**14**], the tangent space $T_{\Phi_i(x_0,t)}\Phi_i(\cdot, t)(V)$ contains the linking line spanned by $L_i$ only in the case that the matrix $I - \{2(1-t)r_i + (2t-1)\ell_i\}(S_i)_{\mathbf{v}}$

is singular. Since this is ruled out by (6.2), $T_{\Phi_i(x_0,t)}\Phi_i(\cdot,t)(V)$ will be transverse to $L_i(x_0) = \ell_i(x_0)\boldsymbol{u}_i(x_0)$, proving statement 3. $\square$

The linking flow extends smoothly to the closure of the smooth stratum of $\mathscr{S}^{\widetilde{M}_i}$.

**Corollary 6.2.3.** *Suppose $x_0 \notin \partial M_i \setminus M_i^\infty$ but $(x_0, L_i)$ does not belong to the smooth stratum in the refined stratification $\mathscr{S}^{\widetilde{M}_i}$. Provided* (6.2) *holds, statements 1–3 in Proposition 6.2.2 are satisfied for a smooth value of $L_i$ on a neighborhood $V$ of $x_0$ within a local manifold component of $x_0$.*

Finally, we derive the local nonsingularity of the linking flow for points $x_0 \in \partial M_i \backslash M_i^\infty$.

**Proposition 6.2.4.** *Let $L_i = \ell_i \boldsymbol{u}_i$ be a smooth value of the linking vector field on a neighborhood $V$ of a point $x_0 \in \partial M_i \setminus M_i^\infty$ (or on a local edge manifold component if $x_0$ is an edge closure point). Assume that the edge shape operator $(S_i)_E$ satisfies the following* edge linking condition *on $V$:*

$$\ell_i < min\left\{\frac{1}{\kappa_{E\,j}}\right\} \text{ for all positive principal edge curvatures } \kappa_{E\,j} \text{ of } (S_i)_E. \tag{6.7}$$

*Then:*

*(1) $\Phi_i : V \times (1/2, 1] \to \mathbb{R}^{n+1}$ is a local diffeomorphism at $(x_0, t)$.*
*(2) $\Phi_i(\cdot, t) : V \to \mathbb{R}^{n+1}$ is a local embedding at $x_0$ for $1/2 < t \leq 1$.*
*(3) $\Phi_i(\cdot, t)(V)$ is transverse to the linking line spanned by $L_i$ for $1/2 < t \leq 1$.*

Proof. In this case, the proof closely follows the proof of Proposition 4.4 in [**14**].

Using the methods in Proposition 6.2.2, we may obtain the transpose Jacobian matrix of $\Phi_i$ written with respect to the edge coordinate basis $\{\frac{\partial}{\partial t}, v_1, \ldots, v_n\}$ about $x_0$ and the basis $\{\boldsymbol{u}_i, v_1, \ldots, v_{n-1}, n\}$ for $\mathbb{R}^{n+1}$:

$$\left(\begin{array}{c|c} 2(\ell_i - r_i) & 0 \\ \hline * & (I_{n-1,1} - [2(1-t)r_i + (2t-1)\ell_i](S_i)_{E\,\mathbf{v}})^T \end{array}\right),$$

where $I_{n-1,1}$ denotes the $n \times n$ matrix with 1's in the first $n-1$ diagonal positions and 0 in the last. The matrix $I_{n-1,1} - \{2(1-t)r_i + (2t-1)\ell_i\}(S_i)_{E\mathbf{v}}$ equals

$$-\{2(1-t)r_i + (2t-1)\ell_i\}\left((S_i)_{E\mathbf{v}} - \frac{1}{2(1-t)r_i + (2t-1)\ell_i}\, I_{n-1,1}\right),$$

which is nonsingular precisely when $\dfrac{1}{2(1-t)r_i + (2t-1)\ell_i}$ is not a generalized eigenvalue of $(S_i)_{E\mathbf{v}}$ for $1/2 < t \leq 1$. By the edge linking condition (6.7), it follows that the above matrix has full rank. The proposition now concludes as in Proposition 6.2.2. □

We utilize the nonsingularity of the grassfire flow in the complement in order to prove the following important proposition.

**Proposition 6.2.5.** $(M_i, L_i)$ *satisfies the radial linking condition* (6.2) *and the edge linking condition* (6.7) *as the Blum medial axis and linking vector field of* $\Omega_i \subset \mathbb{R}^{n+1}$.

PROOF. The proof is very similar to the proof of Proposition 4.6 in [**14**].

Let $V$ be a neighborhood of a point $x_0 \in M_i \setminus M_i^{\infty}$, with $(x_0, L_i)$ in the smooth stratum of the refined stratification and with a smooth value of the linking vector field $L_i$ defined on $V$. (If $(x_0, L_i)$ belongs to a singular stratum of $\mathscr{S}^{\widetilde{M}_i}$, one may extend a local component of $x_0$ in a neighborhood $V$ so that $x_0$ is a point in the interior.) The distance from a point $x_0 + r_i(x_0)\boldsymbol{u}_i(x_0)$ in $\mathcal{B}_i$ to a point $x_0 + (2(1-t)r_i(x_0) + (2t-1)\ell_i(x_0))\boldsymbol{u}_i(x_0)$ in the complement is given by $(2t-1)(\ell_i(x_0) - r_i(x_0))\boldsymbol{u}_i(x_0)$ for $1/2 < t \leq 1$. At time $t' = (2t-1)(\ell_i(x_0) - r_i(x_0))$, the grassfire flow in the complement will consist of points satisfying the condition that their distance from the boundary $\mathcal{B}_i$ along $L_i$ is $(2t-1)(\ell_i(x_0) - r_i(x_0))$. Therefore, using local coordinates on $V$, we may represent the grassfire flow in the complement as

$$\begin{aligned} g(x,t) &= x + r_i(x)\boldsymbol{u}_i(x) + (2t-1)(\ell_i(x_0) - r_i(x_0))\boldsymbol{u}_i(x) \\ &= x + (r_i(x) + (2t-1)\ell_i(x_0) - (2t-1)r_i(x_0))\boldsymbol{u}_i(x). \end{aligned}$$

Thus, its derivative is computed to be

$$\frac{\partial g}{\partial v_k} = v_k + \frac{\partial r_i}{\partial v_k} \cdot \boldsymbol{u}_i + (r_i + (2t-1)\ell_i(x_0) - (2t-1)r_i(x_0)) \cdot \frac{\partial \boldsymbol{u}_i}{\partial v_k},$$

and using (6.3), the vector form of the derivative is given by

$$\begin{aligned} \frac{\partial g}{\partial \mathbf{v}} &= (dr_i(\mathbf{v}) + (r_i + (2t-1)\ell_i(x_0) - (2t-1)r_i(x_0))\mathbf{1}) \cdot \boldsymbol{u}_i \\ &\quad + (I - (r_i + (2t-1)\ell_i(x_0) - (2t-1)r_i(x_0) \cdot (S_i)_{\mathbf{v}})^T \cdot \mathbf{v}, \end{aligned}$$

where $\mathbf{1}$ is a column vector with all entries equal to 1. Since $\dfrac{\partial g}{\partial t'} = \boldsymbol{u}_i$, we see that the transpose Jacobian matrix written with respect to the bases $\{\partial/\partial t', v_1, \ldots, v_n\}$ and $\{\boldsymbol{u}_i, v_1, \ldots, v_n\}$ in the source and target, respectively, is given by

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline (r_i + (2t-1)\ell_i(x_0) - (2t-1)r_i(x_0))\mathbf{1} & (I - (r_i + (2t-1)\ell_i(x_0) - \\ +dr_i(\mathbf{v}) & (2t-1)r_i(x_0)) \cdot (S_i)_{\mathbf{v}})^T \end{array}\right).$$

At the point $x_0$, this matrix becomes

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline dr_i(\mathbf{v}) + (2(1-t)r_i(x_0) + & (I - (2(1-t)r_i(x_0) + \\ (2t-1)\ell_i(x_0))\mathbf{1} & (2t-1)\ell_i(x_0)) \cdot (S_i)_{\mathbf{v}})^T \end{array}\right),$$

which is nonsingular if and only if $\dfrac{1}{2(1-t)r_i(x_0) + (2t-1)\ell_i(x_0)}$ is not an eigenvalue of $(S_i)_{\mathrm{rad}}$. Since the grassfire flow in the complement is nonsingular, we conclude that $\dfrac{1}{2(1-t)r_i(x_0) + (2t-1)\ell_i(x_0)}$ is not an eigenvalue for $1/2 < t \leq 1$, i.e., for values in the interval $\left[\dfrac{1}{\ell_i(x_0)}, \dfrac{1}{r_i(x_0)}\right)$. Since the radial curvature condition ensures that all eigenvalues are less than $\dfrac{1}{r_i(x_0)}$, this proves that the radial linking condition (6.2) holds. The proof that the edge linking condition (6.7) holds is analogous. $\square$

## 6.3. Geometry in the complement from the linking structure

In this section, we show how the linking flow enables one to obtain a matrix representation $(S_0)_{\boldsymbol{v}'}$ of the radial shape operator $(S_0)_{\text{rad}}$ on the linking medial axis $M_0$ in terms of a matrix representation $(S_i)_{\boldsymbol{v}}$ of the radial shape operator $(S_i)_{\text{rad}}$ on $M_i$ for some $i$. This enables one to obtain the geometry in the complement directly from the linking structure.

The proof of the following proposition closely follows the proof of Proposition 2.1 in [**15**] with minor changes.

**Proposition 6.3.1.** *Choose a smooth value of $L_i$ on a neighborhood $V$ of a smooth point $x_0 \in M_i \backslash M_i^{\infty}$ with respect to the refined stratification, and suppose $\frac{1}{\ell_i}$ is not an eigenvalue of $(S_i)_{rad}$ at $x_0$. Let $x_0' = \Phi_i(x_0, 1)$, let $\{v_1, \dots, v_n\}$ be a basis for $T_{x_0}M_i$, and for all $j$, let $\{v_j'\}$ denote the image of $\{v_j\}$ under $d\Phi_i(x_0, 1)$. If $(S_i)_{\boldsymbol{v}}$ is a matrix representation for $(S_i)_{rad}$ and $\boldsymbol{v}'$ denotes the image of $\boldsymbol{v}$ in vector form, then the radial shape operator $(S_0)_{rad}$ of the linking medial axis $M_0$ at $x_0'$ has a matrix representation with respect to $\boldsymbol{v}'$ given by*

$$(S_0)_{\boldsymbol{v}'} = -(I - \ell_i \cdot (S_i)_{\boldsymbol{v}})^{-1} (S_i)_{\boldsymbol{v}}. \tag{6.8}$$

Proof. Let $\boldsymbol{v}'$ be the column vector with $i-$th entry $v_i'$. One may easily verify that $\frac{\partial \boldsymbol{u}_i}{\partial v_i'} = \frac{\partial \boldsymbol{u}_i}{\partial v_i}$ (see the proof of Proposition 2.1 in [**15**]). From (6.3), we know that

$$\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{v}} = (A_i)_{\boldsymbol{v}} \cdot \boldsymbol{u}_i - (S_i)_{\boldsymbol{v}}^T \cdot \boldsymbol{v}. \tag{6.9}$$

Using (6.4), we obtain

$$\boldsymbol{v}' = \frac{\partial \Phi_i(\cdot, 1)}{\partial \boldsymbol{v}} = (d\ell_i(\boldsymbol{v}) + \ell_i \cdot (A_i)_{\boldsymbol{v}}) \cdot \boldsymbol{u}_i + (I - \ell_i \cdot (S_i)_{\mathbf{v}})^T \cdot \mathbf{v}. \tag{6.10}$$

By assumption, $(I - \ell_i \cdot (S_i)_{\mathbf{v}})^T$ is nonsingular, and we may solve for $\boldsymbol{v}$ in (6.10). Since $\dfrac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{v}'} = \dfrac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{v}}$, (6.9) becomes

$$\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{v}'} = A'_{\boldsymbol{v}'} \cdot \boldsymbol{u}_i - (I - \ell_i \cdot (S_i)^T_{\boldsymbol{v}})^{-1} \cdot (S_i)^T_{\boldsymbol{v}} \boldsymbol{v}' \tag{6.11}$$

with $A'_{\boldsymbol{v}'} = (A_i)_{\boldsymbol{v}} - (I - \ell_i \cdot (S_i)^T_{\boldsymbol{v}})^{-1} \cdot (S_i)^T_{\boldsymbol{v}} (d\ell_i(\boldsymbol{v}) + \ell_i \cdot (A_i)_{\boldsymbol{v}})$. Then the definition of the radial shape operator implies that, after the projection onto $T_{x'_0} M_0$ and taking transposes,

$$(S_0)_{\boldsymbol{v}'} = -(I - \ell_i \cdot (S_i)_{\boldsymbol{v}})^{-1} (S_i)_{\boldsymbol{v}}. \tag{6.12}$$

The negative sign is needed since $\boldsymbol{u}_i$ points in the opposite direction as the unit radial vector at $x'_0 \in M_0$. $\square$

### 6.4. Integration over regions in the complement

Let $\Gamma \subset \mathbb{R}^{n+1}$ be a bounded Borel measurable region that intersects $\Omega^C$, the region complementary to $\Omega = \coprod_{i=1}^{q} \Omega_i$, and possibly one or more of the $\Omega_i$, as illustrated in Figure 6.1. We also assume that $\Gamma$ does not intersect $(\Omega^C)^\infty$, which is the region outside of $\Omega$ within which no linking occurs. In this section, we show how to integrate a Borel measurable and Lebesgue integrable function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ over $\Gamma$ as a sum of integrals over the individual medial axes $\coprod_{i=1}^{q} M_i$. The computations given in Section 2.5 in Chapter 2 for integration over a single medial axis have natural extensions in the multi–region setting using the linking functions.

The Blum medial linking structure yields a decomposition of the complement into *linking neighborhoods* as follows.

**Definition 6.4.1.** *For fixed $i, j \in \{1, \ldots, q\}$, the $j$–**th linking neighborhood of** $M_i$**,** denoted $\mathcal{R}_{i \to j}$, is the region in $\mathbb{R}^{n+1}$ given by*

$$R_{i \to j} = \{x + tL_i(x) : x \in M_i \backslash M_i^\infty, 0 \le t \le 1, \text{ and } \exists w \in M_j \text{ with } x + L_i(x) = w + L_j(w)\}.$$

*The **linking neighborhood of** $M_i$ is the region $\mathcal{R}_i = \bigcup_{j=1}^{q} \mathcal{R}_{i \to j}$.*

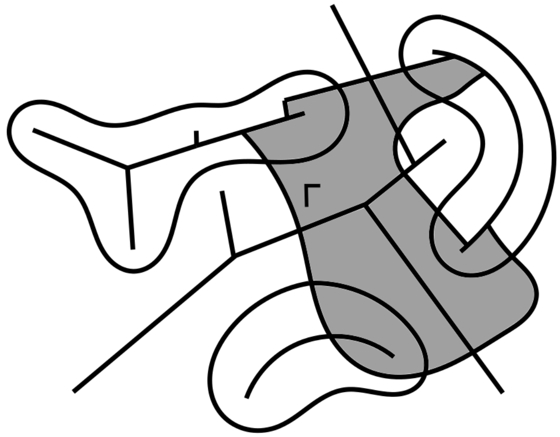


FIGURE 6.1. *Example of a region $\Gamma$ over which one may integrate.*

The proof of the following theorem is similar to the proofs of Theorem 6 and Corollary 7 in [**13**], which establish how to integrate a function on a region $\Gamma \subset \Omega_i$ as an integral over its medial axis. (See Theorems 2.5.4 and 2.5.5 in Chapter 2. Here, $\Gamma = \bigcup_i \Gamma_i \subset \mathbb{R}^{n+1}$ is a bounded Borel measurable region with $\Gamma_i \subset \mathcal{R}_i$, so that $\Gamma$ intersects $\Omega^C \setminus (\Omega^C)^\infty$ as well as the interior of one or more $\Omega_i$, with the only overlaps being along the linking medial axis.

**Theorem 6.4.2.** *Let $\Omega = \coprod_{i=1}^{q} \Omega_i$ be defined by a generic embedding $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$, and let $\Gamma = \bigcup_i \Gamma_i \subset \mathbb{R}^{n+1}$ be a bounded Borel measurable region as above. Let $g : \Gamma \to \mathbb{R}$ be a Borel measurable and Lebesgue integrable function, and for each $i = 1, \ldots, q$, let*

$$\widetilde{g}_i = \int_0^1 \chi_{\Gamma_i} \cdot g(x + t\ell_i(x)) \cdot \det(I - t\ell_i(S_i)_{rad})\, dt. \tag{6.13}$$

*Then for every $i$, $\widetilde{g}_i$ is defined for almost all $x \in \widetilde{M}_i$, integrable on $\widetilde{M}_i$, and*

$$\int_\Gamma g\, dV = \sum_{i=1}^{q} \int_{\widetilde{M}_i} \widetilde{g}_i \cdot \ell_i\, dM_i. \tag{6.14}$$

PROOF. The proof that every $\widetilde{g}_i$ is defined and integrable follows the same reasoning as in the proof of Theorem 6 in [**13**]. We shall show that, for each $i$,

$$\int_{\Gamma_i} g\, dV = \int_{\widetilde{M}_i} \widetilde{g}_i \cdot \ell_i \, dM_i \tag{6.15}$$

by utilizing the linking flow. As in [**13**], we reduce to the case of showing that (6.15) holds on a single compact manifold $M_{i_j}$ with boundaries and corners with interior consisting of points in $(M_i)_{\text{reg}}$ and with a smooth value of the linking vector field $L_i$ defined on $M_{i_j}$. Then the linking flow $\Phi_i(\cdot, t)$ restricted to $M_{i_j}$ is a diffeomorphism on the interior for $0 \leq t \leq 1/2$ and for $1/2 < t \leq 1$. For the first half of the flow, note that a basic change of variables shows that

$$\widetilde{g}_i^{(1)} := 2\int_0^{1/2} \chi_{\Gamma_i \cap \Omega_i} \cdot (g \circ \Phi_i)(x,t) \cdot \det(I - 2tr_i(S_i)_{\text{rad}})\, dt \tag{6.16}$$

equals the Crofton-type formula for integration given in (2.5.5) in Chapter 2. Then

$$\int_{\Gamma_i \cap \Omega_i} g\, dV = \int_{M_{i_j}} \widetilde{g}_i^{(1)} \cdot r_i \, dM_i.$$

For the second half of the flow, let $\{v_1, \ldots, v_n\}$ be a basis for $T_{x_0} M_{i_j}$, let

$$C_i = (I - [2(1-t)r_i + (2t-1)\ell_i](S_i)_{\mathbf{v}})^T,$$

and denote by $\widetilde{C}_i$ the linear transformation that maps $v_j$ to $\displaystyle\sum_{k=1}^{n} c_{kj} v_k$. Recall the medial measure $\rho_i = \langle \boldsymbol{u}_i, \boldsymbol{n}_i \rangle$ on $M_i$. Using the fact that $\dfrac{\partial \Phi_i}{\partial t} = 2(\ell_i - r_i)\boldsymbol{u}_i$ for $1/2 < t \leq 1$, we obtain by the change of variables formula

$$
\begin{aligned}
\Phi_i^* dV(\frac{\partial}{\partial t}, v_1, \ldots, v_n) =& \det(2(\ell_i - r_i)\boldsymbol{u}_i, d\Phi_i(v_1), \ldots, d\Phi_i(v_n)) \\
=& 2(\ell_i - r_i) \cdot \det(\boldsymbol{u}_i, \widetilde{C}_i(v_1), \ldots, \widetilde{C}_i(v_n)) \\
=& 2(\ell_i - r_i) \cdot \det(\langle \boldsymbol{u}_i, \boldsymbol{n}_i \rangle \cdot \boldsymbol{n}_i, \widetilde{C}_i(v_1), \ldots, \widetilde{C}_i(v_n)) \\
=& 2(\ell_i - r_i) \cdot \det(C_i) \cdot \rho_i \cdot \det(\boldsymbol{n}_i, v_1, \ldots, v_n) \\
=& 2(\ell_i - r_i) \cdot \det\left[I - [2(1-t)r_i + (2t-1)\ell_i](S_i)_{\mathrm{rad}}\right] dM_i(v_1, \ldots, v_n).
\end{aligned}
$$

If we let

$$
(6.17) \quad \widetilde{g}_i^{(2)} := 2 \int_{1/2}^{1} \chi_{\Gamma_i \setminus \Omega_i} \cdot (g \circ \Phi_i)(x,t) \cdot \det(I - (2(1-t)r_i + (2t-1)\ell_i)(S_i)_{\mathrm{rad}})\, dt,
$$

Fubini's Theorem then implies that

$$
\int_{\Gamma_i \setminus \Omega_i} g\, dV = \int_{M_{i_j}} \widetilde{g}_i^{(2)} \cdot (\ell_i - r_i)\, dM_i.
$$

Using the change of variables $t' = 2tr_i$ for $\widetilde{g}_i^{(1)}$ in (6.16), and $t' = 2(1-t)r_i + (2t-1)\ell_i$ for $\widetilde{g}_i^{(2)}$ in (6.17), we may write

$$
\begin{aligned}
\widetilde{g}_i^{(1)} \cdot r_i &= \int_0^{r_i} \chi_{\Gamma_i \cap \Omega_i} \cdot g(x + t\boldsymbol{u}_i(x)) \cdot \det(I - t\,(S_i)_{\mathrm{rad}})\, dt, \\
\widetilde{g}_i^{(2)} \cdot (\ell_i - r_i) &= \int_{r_i}^{\ell_i} \chi_{\Gamma_i \setminus \Omega_i} \cdot g(x + t\boldsymbol{u}_i(x)) \cdot \det(I - t\,(S_i)_{\mathrm{rad}})\, dt.
\end{aligned}
$$

Thus,

$$
(6.18) \qquad \widetilde{g}_i^{(1)} \cdot r_i + \widetilde{g}_i^{(2)} \cdot (\ell_i - r_i) = \int_0^{\ell_i} \chi_{\Gamma_i} \cdot g(x + t\boldsymbol{u}_i(x)) \cdot \det(I - t\,(S_i)_{\mathrm{rad}})\, dt.
$$

After a second change of variables, the integral in (6.18) becomes $\widetilde{g}_i \cdot \ell_i$ with $\widetilde{g}_i$ as in (6.13).

Therefore, using the fact that the linking medial axis has measure 0 in $\mathbb{R}^{n+1}$, we obtain

$$\begin{aligned}
\int_{\Gamma} g\, dV &= \sum_{i=1}^{q} \int_{\Gamma_i} g\, dV \\
&= \sum_{i=1}^{q} \left( \int_{\widetilde{M}_i} \widetilde{g}_i^{(1)} \cdot r_i\, dM_i + \int_{\widetilde{M}_i} \widetilde{g}_i^{(2)} \cdot (\ell_i - r_i)\, dM_i \right) \\
&= \sum_{i=1}^{q} \int_{\widetilde{M}_i} \widetilde{g}_i \cdot \ell_i\, dM_i.
\end{aligned}$$

□

## 6.5. Measures of comparison for a collection of regions

In this section, we introduce geometric invariants that measure two specific aspects of the positional geometry of a collection of regions: namely, closeness and significance. The invariants, which are computed directly from the Blum medial linking structure, depend on a notion of closure of the linking structure, and we address this issue first in Section 6.5.1.

**6.5.1. Developing a bounded version of the medial linking structure.** Up to this point, we have viewed the medial linking structure as an infinite structure since the linking medial axis extends indefinitely in the complement. However, the essential linking information captured by a collection of linking neighborhoods is concentrated in particular areas between the regions. Linking between two regions may naturally come to an end due to the introduction of one or more other regions. For example, in Figure 6.4, linking between the regions with medial axes $M_2$ and $M_3$ naturally ends due to the presence of the regions with medial axes $M_1$ and $M_4$. However, there will not be a natural end to linking for every pair of regions within a given collection, and we require a rigorous means of obtaining a bounded version of the linking structure.

To this end, assume $\Delta \subset \mathbb{R}^{n+1}$ is an ambient region such that $\Omega = \bigcup_{i=1}^{q} \Omega_i \subset \Delta$. Then the intersection $\partial\Delta \cap M_0$ determines where the linking vector fields should terminate, so we restrict the $j-$th linking neighborhood to $R_{i \to j} := \mathcal{R}_{i \to j} \cap \Delta$, with $R_i = \bigcup_{j=1}^{q} R_{i \to j}$. For example, possible choices of enclosing region $\Delta$ are the smallest sphere containing $\Omega$, the convex hull of $\Omega$, a threshold on $\ell_i$ or $\ell_i - r_i$, or an enclosing region resulting from other considerations such as physiology of the boundary for medical images.

In what follows, we shall restrict the linking neighborhood $R_i$ of $M_i$ for every $i$ to the region $R_i \setminus R_{i \to i}$, so that self-linking does not contribute to the linking neighborhoods.

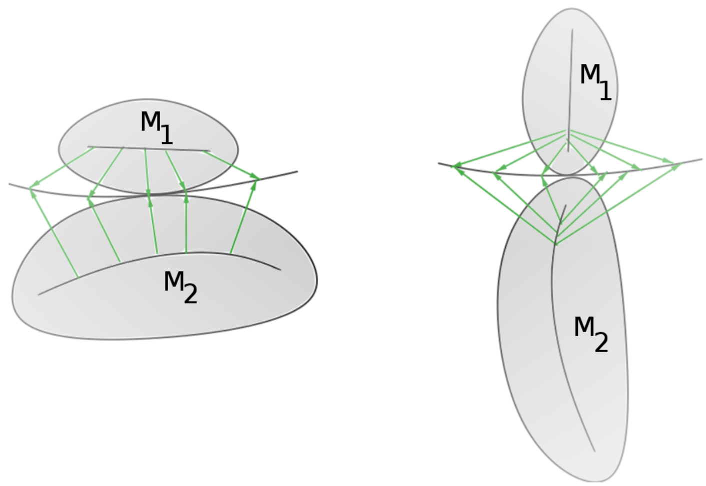


FIGURE 6.2. *Illustration for measuring closeness.*
*For each pair of regions in Figure 6.2, compare the areas of the portions of the linking neighborhoods inside the regions to the combined total areas of the linking neighborhoods. For the pair of regions on the left, the areas inside the regions sum to a higher percentage of the areas of the linking neighborhoods than do the corresponding areas for the pair of regions on the right. Hence, from the volumetric perspective, the pair of regions on the left exhibit a higher degree of closeness.*

**6.5.2. Distance vs. volumetric approaches to closeness and significance.** In this section, we discuss how distance measures of closeness and significance compare to volumetric measures of these quantities.

Recall from Section 2.6 in Chapter 2 that one objective in multi–region shape analysis is to measure proximity of regions to one another. In Figures 6.3 and 6.4, a comparison of the minimum distances between the region with medial axis $M_1$ and the other regions yields higher measures of closeness in Figure 6.4 than in Figure 6.3. However, the notion of closeness of regions involves more than simply the minimum distance between two regions. In Figure 6.2, since both pairs of regions touch one another, they exhibit the same degree of closeness as measured by the minimum distance criterion. However, this notion fails to capture the fact that for the regions on the left, larger portions of the regions are close and there is less space between the regions. Thus, to adequately measure closeness, an appropriate measure captures volumetric aspects of portions of the regions and their complement.

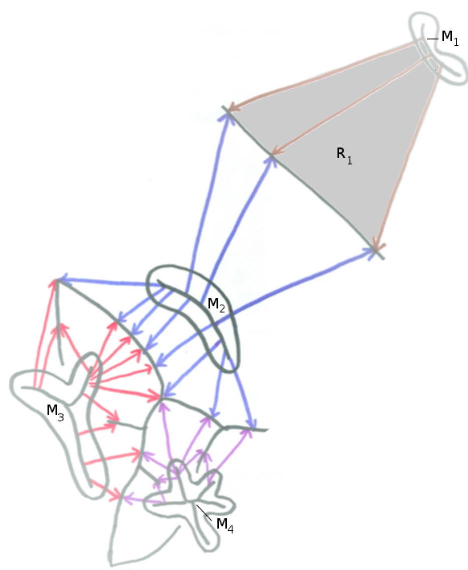


FIGURE 6.3. *Illustration of relative significance of regions. A small positional change of the region with medial axis $M_1$ would have little to no impact on the other regions in the configuration. The area of the portion of the linking neighborhood inside the region is a very small percentage of the total area of $R_1$.*

The notion of significance of a particular region within a collection of regions encompasses multiple factors, one of which is proximity to other regions. However, a purely distance–based measure of significance fails to capture all other aspects of the positional geometry of a collection of regions. For instance, another factor impacting significance is the size of the region $\Omega_i \subset \mathbb{R}^{n+1}$. Recall that $\mathrm{vol}(\Omega_i)$ may be computed as a medial integral (see Section 2.5.4 in Chapter 2). The inadequacy of using $\mathrm{vol}(\Omega_i)$ to measure significance lies in the fact that individual region size within a configuration is not meaningful unless it is considered in the context of additional factors, such as the sizes of the other regions, their proximity to $\Omega_i$, and their positioning relative to one another. An

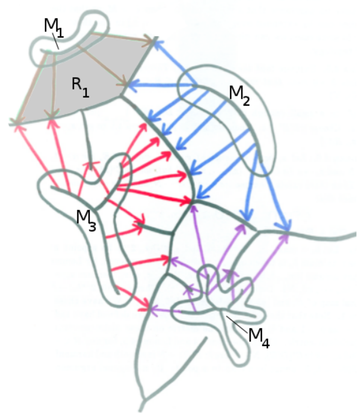


FIGURE 6.4. *Example of how a change in positioning affects significance. For the region with medial axis $M_1$, a small positional change would also alter the linking neighborhoods of the regions with medial axes $M_2$ and $M_3$. For this region, the value of its significance measure is higher in Figure 6.4 than in Figure 6.3 as the portion of the linking neighborhood $R_1$ inside the region forms a higher percentage of the total area of $R_1$ in Figure 6.4.*

appropriate volumetric measure of significance of a region should capture how a small positional change of the region impacts the overall configuration. For example, in Figure 6.3, if the region with medial axis $M_1$ were translated or rotated slightly, this would have a small effect on $R_{1\to 2}$, its linking neighborhood with the region with medial axis $M_2$, but would otherwise have no effect on the other regions in the configuration. On the other hand, a small translation or rotation of the same region in Figure 6.4 would produce a greater alteration of the linking neighborhood of the region with medial axis $M_2$ and would also impact the linking neighborhood of the region with medial axis $M_3$.

In the next section, we introduce invariants of both closeness and significance which take values between 0 and 1 and which are defined in terms of volumes. These invariants may be computed using the medial linking structure.

**6.5.3. Measuring closeness and significance of regions from the linking structure.** For a pair of regions $\Omega_i$ and $\Omega_j$, we define the following measure of closeness, which we shall denote by $C$:

$$C(\Omega_i, \Omega_j) = \frac{\mathrm{vol}(\Omega_i \cap R_{i\to j}) + \mathrm{vol}(\Omega_j \cap R_{j\to i})}{\mathrm{vol}(R_{i\to j}) + \mathrm{vol}(R_{j\to i})}. \tag{6.19}$$

For the region $\Omega_i$, we define a significance measure, which we shall denote by $S$, as follows:

$$S(\Omega_i) = \frac{\mathrm{vol}(\Omega_i \cap R_i)}{\mathrm{vol}(R_i)}. \tag{6.20}$$

**Remark 6.5.1.** For the case that $\Omega = \Omega_1 \bigcup \Omega_2$, there is a relationship between the two comparison measures. Since the linking neighborhoods satisfy $R_{1\to 2} = R_1$ and $R_{2\to 1} = R_2$, the value of $C(\Omega_1, \Omega_2)$ will be bounded above by the sum of the values of $S$ for each region, i.e.,

$$\begin{aligned} C(\Omega_1, \Omega_2) &= \frac{vol(\Omega_1 \cap R_{1\to 2}) + vol(\Omega_2 \cap R_{2\to 1})}{vol(R_{1\to 2}) + vol(R_{2\to 1})} \\ &= \frac{vol(\Omega_1 \cap R_1) + vol(\Omega_2 \cap R_2)}{vol(R_1) + vol(R_2)} \\ &< \frac{vol(\Omega_1 \cap R_1)}{vol(R_1)} + \frac{vol(\Omega_2 \cap R_2)}{vol(R_2)} \\ &= S(\Omega_1) + S(\Omega_2). \end{aligned} \tag{6.21}$$

We can combine these two measures to yield a "tiered graph." Each region $\Omega_i$ represents a vertex of the graph, with each vertex weighted by the significance measure $S(\Omega_i)$. There is an edge between two vertices if the regions to which they correspond are linked. The edge between the vertices representing $\Omega_i$ and $\Omega_j$ is weighted by the value of $C(\Omega_i, \Omega_j)$. Then the closeness and significance measures may be viewed as defining "height functions" on the graph since they induce multiple "tiers" or levels of comparison among the regions. That is, by first considering the highest values of the comparison measures and successively filtering the graph into subgraphs, one may obtain orderings of the regions based on their varying levels of relative significance and proximity to other regions.

**Remark 6.5.2.** Although not proven in this dissertation, we believe that the measures of closeness and significance are continuous under sufficiently small generic perturbations of the regions, which would provide a stability result for the graph structure.

We obtain the following proposition for the application of the linking structure to multi–region shape analysis.

**Proposition 6.5.3.** *For $n \leq 6$, let $\phi \in \mathrm{DEmb}(X, \mathbb{R}^{n+1})$ be a generic embedding defining a Blum medial linking structure associated to regions $\{\Omega_i\}_{i=1}^{q}$. The closeness and significance of the regions and the tiered graph can be computed directly from the linking structure using the integral formulas* (6.24) *and* (6.25)*.*

Proof. Define the following multivalued functions on $M_i$:

$$c_i = \int_0^1 \chi_{\Omega_i \cap R_{i\to j}} \cdot \det(I - t\, r_i\, (S_i)_{\mathrm{rad}})\, dt, \tag{6.22}$$

$$d_i = \int_0^1 \chi_{R_{i\to j}} \cdot \det(I - t\ell_i\, (S_i)_{\mathrm{rad}})\, dt. \tag{6.23}$$

Thus, $c_i$ is a line integral over the portion of the radial line within $\Omega_i \cap R_{i\to j}$, while $d_i$ is a line integral over the portion of the linking line within the entire $j-$th linking neighborhood $R_{i\to j}$. We may then integrate these functions over the medial axis, as $M_i$ parametrizes such lines. Therefore, the formulas for $C(\Omega_i, \Omega_j)$ and $S(\Omega_i)$ are

$$C(\Omega_i, \Omega_j) = \frac{\displaystyle\int_{\widetilde{M}_i} c_i \cdot r_i\, dM_i + \int_{\widetilde{M}_j} c_j \cdot r_j\, dM_j}{\displaystyle\int_{\widetilde{M}_i} d_i \cdot \ell_i\, dM_i + \int_{\widetilde{M}_j} d_j \cdot \ell_j\, dM_j} \tag{6.24}$$

and

$$S(\Omega_i) = \frac{\displaystyle\int_{\widetilde{M}_i} c_i \cdot r_i\, dM_i}{\displaystyle\int_{\widetilde{M}_i} d_i \cdot \ell_i\, dM_i}. \tag{6.25}$$

$\square$

## 6.6. Future directions

We conclude with a brief discussion of some future extensions involving the Blum medial linking structure.

Throughout this dissertation, we assumed that a generic embedding defines a collection of $q$ disjoint regions. Eventually, we wish to remove the restriction that the regions be disjoint and allow for the possibility of regions to be tangent along a portion of their boundaries or lie within other regions. This is a more practical assumption from the perspective of medical image analysis, and has the potential to be of use in applications involving the presence of tumors inside organs.

In addition, it is desirable from the perspective of medical image analysis to relax the conditions on the Blum medial axis of a region and allow for more general medial–type structures, known as *skeletal structures*; see [**14**]. Like the Blum medial axis and radial vector field, a skeletal structure consists of a Whitney stratified set $M$ and a multivalued radial vector field $U$ defined on $M$ consisting of vectors that point from $M$ to the corresponding sphere tangency points on the boundary. However, the radial vector field of a skeletal structure need not satisfy all of the special properties that are satisfied in the Blum case; for instance, radial vectors based at the point $x_0 \in M$ may have different lengths, may not be orthogonal to the boundary, and they need not make the same angle with $T_{x_0}M$. Damon is extending the theory of the linking structure for the situation in which some or all of the elements in a given collection are defined by skeletal structures, rather than regions with Blum medial axes.

Furthermore, Pizer is beginning the process of implementing the linking structure for use in shape analysis involving configurations of objects in medical images. Just as a medial representation of a single object has proven to be an effective tool in medical imaging, we hope that the natural extensions of medial analysis which the linking structure provides will prove beneficial in the area of multi–object shape analysis.

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