DRAFT VERSION OF METHOD NOT YET TESTED ON REALISTIC MEDICAL OBJECT VARIATION.

LORENTZIAN GEODESIC FLOWS BETWEEN HYPERSURFACES IN EUCLIDEAN SPACES

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VERY PRELIMINARY VERSION

Introduction

We consider the problem of constructing a natural diffeomorphic flow between hypersurfaces M_0 and M_1 of $\mathbb{R}^n$ which is in some sense both "natural" and "geodesic" viewed in some appropriate space (as in figure).

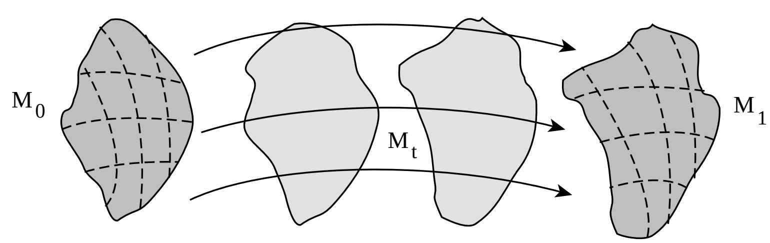

FIGURE 1. Diffeomorphic Flow between hypersurfaces of Euclidean space induced by a "Geodesic Flow" in an associated space

There are several approaches to this question. One is from the perspective of a Riemannian metric on the group of diffeomorphisms of $\mathbb{R}^n$. If the smooth hypersurfaces M_i bound compact regions Ω_i , then the group of diffeomorphisms $Diff(\mathbb{R}^n)$ acts on such regions Ω_i and their boundaries. Then, if $\varphi_t, 1 \leq t \leq 1$, is a geodesic in $Diff(\mathbb{R}^n)$ beginning at the identity, then $\varphi_t(\Omega)$ (or $\varphi_t(M_i)$) provides a path interpolating between $\Omega_0 = \varphi_0(\Omega) = \Omega$ and $\Omega_1 = \varphi_1(\Omega)$. Then, the geodesic equations can be computed and numerically solved to construct the flow φ_t. This is the method developed by Younes, Trouve, Glaunes [Tr], [YTG], [BMTY], [YTG2], and Mumford, Michor [MM], [MM2] etc.

An alternate approach which we consider in this paper requires that we are given a correspondence between M_0 and M_1, defined by a diffeomorphism $\chi : M_0 \to M_1$, which need not be the restriction of a global diffeomorphism of $\mathbb{R}^n$ (and the M_i may have boundaries). Then, if we map M_0 and M_1 to submanifolds of a natural ambient space Λ, we can seek a geodesic flow between M_0 and M_1, viewed as

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submanifolds of Λ, sending x to $\varphi(x)$ along a geodesic. Then, we use this geodesic flow to define a flow between M_0 and M_1 back in $\mathbb{R}^n$.

The simplest example of this is the "radial flow" from M_0 using the vector field U on M_0 defined by $U(x) = \varphi(x) - x$. Then, the radial flow is the geodesic flow in $\mathbb{R}^n$ defined by $\varphi_t(x) = x + t \cdot U(x)$. The analysis of the nonsingularity of the radial flow is given in [D1] in the more general context of "skeletal structures". This includes the case where M_1 is a "generalized offset surface" of M_0 via the generalized offset vector field U.

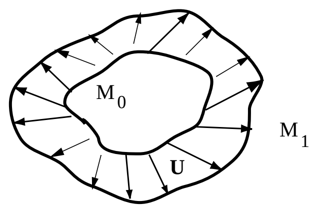

FIGURE 2. Hypersurface M_0 and radial vector field U define a generalized offset surface M_1 obtained from a radial flow of the skeletal structure (M_0, U). This is a "Geodesic Flow" in $\mathbb{R}^n$.

In this paper, we give an alternate approach to interpolation between hypersurfaces with a given correspondence. While the radial flow views the hypersurface as a collection of points, we will instead view it as defined by the collection of tangent spaces. This leads to consideration of geodesic flows between "dual varieties". However, the dual varieties tradtionally lie in the "dual projective space". it has a natural Riemannian metric; however, the geodesic flow for this metric does not have certain natural properties (such as invariance under translation) that are desirable. Instead, we shall define a 'Lorentzian map" to a Lorentzian space Λ and represent the "dual varieties" as subspaces of the Lorentzian space Λ. Then, we use the geodesic flow for the Lorentzian metric on Λ, and then transform that geodesic flow back to a flow between the original manifolds in $\mathbb{R}^n$.

We shall give conditions that the resulting flow is nonsingular. Furthermore, we deduce the form of the flow when M_1 is obtained from M_0 by standard transformations of $\mathbb{R}^n$, exhibiting them as appropriate geodesic flows.

This applies even to the case of manifolds with boundaries and provides an answer to a question posed by Stephen Pizer for medial surfaces of regions in $\mathbb{R}^3$. He asked whether there is a natural flow between surfaces in $\mathbb{R}^3$, which is defined in terms of the pairs consisting of the points and their surface normals, and which generalizes transformations such as translations, homotheties, and rotations. The flow we define replaces the pair by the tangent plane and then determines a natural geodesic flow on the tangent planes, which does have the desirable properties.

CONTENTS

1. **Overview**

As mentioned in the introduction there are two main methods for deforming one given hypersurface $M_0 \subset \mathbb{R}^n$ to another M_1. One is to find a path ψ_t in G, which is some specified a group of diffeomorphisms of $\mathbb{R}^n$, from the identity so that $\psi_1(M_0) = M_1$ (and $\psi_0(M_0) = M_0$).

Another approach involves constructing a geometric flow between M_0 and M_1. Several flows such as curvature flows do not provide a flow to a specific hypersurface such as M_1. An alternate approach which we shall use will assume that we have a correspondence given by a diffeomorphism $\chi : M_0 \to M_1$ and construct a "geodesic flow"which at time $t = 1$ gives χ. The geodesic flow will be on an associated space $\mathcal{Y}$. We shall consider natural maps $\varphi_i : M_i \to \mathcal{Y}$, where $\mathcal{Y}$ is a distinguished space which reflects certain geometric properties of the M_i.

$$
\begin{array}{ccc} M_0 & \xrightarrow{\varphi_0} & \mathcal{Y} \\ \chi \downarrow & \nearrow_{\varphi_1} & \\ M_1 & & \end{array} \tag{1.1}
$$

Definition 1.1. Given smooth maps $\varphi_i : M_i \to \mathcal{Y}$ and a diffeomorphism $\chi : M_0 \to M_1$ A *geodesic flow* between the maps φ_i is a smooth map $\tilde{\psi}_t : M_0 \times [0, 1] \to \mathcal{Y}$ such that for any $x \in M_0$, $\tilde{\psi}_t(x) : [0, 1] \to \mathcal{Y}$ is a (minimal) geodesic from $\varphi_0(x)$ to $\varphi_1 \circ \chi(x)$

Remark . We shall also refer to the geodesic flow as being between the $\tilde{M}_i = \varphi_i(M_i)$. However, we note that it is possible for more than one $x_i \in M_0$ to map to the same point in $y \in \mathcal{Y}$, however, the geodesic flow from y can differ for each point x_i.

Then, we will complement this with a method for finding the corresponding flow ψ_t between M_0 and M_1 such that $\varphi_t \circ \psi_t = \tilde{\psi}_t$, where $\varphi_t : \psi_t(M_0) \to \tilde{\psi}_t(M_0)$. We furthermore want this flow to satiafy certain properties. The main property is that the flow construction is invariant under the action of the group formed from rigid transformations and homotheties (scalar multiplication). By this we mean: if $M_0' = A(M_0)$ and $M_1' = A(M_1)$ are transforms of M_0 and M_1 by a rigid transformation or homothety A, and M_t is the flow between M_0 and M_1, then $A(M_t)$ gives the flow between M_0' and M_1'. Also, it would be desirable if uniform translations, homotheties, and rotations would also give geodesic flows.

We are specifically interested in a "geodesic flow"which will be a flow defined using the tangent bundles TM_0 to TM_1 so that we specifically control the flow of the tangent spaces. At first, an apparent natural choice is the dual projective space $\mathbb{R}P^{n\vee}$. Via the tangent bundle of a hypersurface $M \subset \mathbb{R}^n$ there is the natural map $\delta : M \to \mathbb{R}P^{n\vee}$, sending $x \mapsto T_xM$. The natural Riemannian structure on the real projective space $\mathbb{R}P^{n\vee}$ is induced from S^n via the natural covering map $S^n \to \mathbb{R}P^n$, so that geodesics of S^n map to geodesics on $\mathbb{R}P^{n\vee}$. However, simple examples show that the induced geodesic slow on $\mathbb{R}P^{n\vee}$ is not invariant under translation in $\mathbb{R}^n$. In fact, this Riemannian geodesic flow between the hyperplanes given by $\mathbf{n} \cdot \mathbf{x} = c_0$ and $\mathbf{n} \cdot \mathbf{x} = c_1$ is given by $\mathbf{n} \cdot \mathbf{x} = c_t$, where $c_t = \tan(t \arctan(c_1) + (1-t) \arctan(c_0))$. It is easily seen that if we translate the two planes by adding a fixed amount d to each c_i, then the corresponding formula does not give the translation of the first.

We will use an alternate space for $\mathcal{Y}$, namely, the Lorentzian space Λ^{n+1} which is a Lorentzian subspace of Minkowski space $\mathbb{R}^{n+2,1}$. In fact the images will be in a special subspace $\mathcal{R} \subset \Lambda^{n+1}$. On Λ^{n+1} it is classical that the geodesics are intersections with planes through the origin in $\mathbb{R}^{n+2,1}$. This allows a simple description of the geodesic flow on Λ^{n+1}. We transfer this flow to a flow on $\mathbb{R}^n$ using an inverse envelope construction, which reduces to solving systems of linear equations. We will give conditions for the smoothness of the inverse construction which uses knowledge of the generic Legendrian singularities.

We shall furthermore see that the construction is invariant under the action of rigid transformations and homotheties. In addition, uniform translations and homotheties will be geodesic flows, and a variant of uniform rotation is also a geodesic flow.

2. **Semi-Riemannian Manifolds and Lorentzian Manifolds**

A *Semi-Riemannian manifold* M is a smooth manifold M, with a nondegenerate bilnear form $< \cdot, \cdot >_x$ on the tangent space T_xM, for eaxh $x \in M$ which smoothly varies with x. We do not require that $< \cdot, \cdot >_x$ be positive definite. We denote the index of $< \cdot, \cdot >_{(x)}$ by ν. In the case that $\nu = 1$, M is referred to as a Lorentzian manifold.

A basic example is Minkowski space which is $\mathbb{R}^{n+1}$ with bilinear form defined for $v = (v_1, \dots, v_{n+1})$ and $w = (w_1, \dots, v_{n+1})$

$$< v, w >_L \quad = \quad \sum_{i=1}^{n} v_i \cdot w_i - v_{n+1} \cdot w_{n+1}$$

There are a number of different notations for Minkowski space. We shall use $\mathbb{R}^{n+1,1}$. We shall also use the notation $< \cdot, \cdot >_L$ for the Lorentzian inner product on $\mathbb{R}^{n+1,1}$.

A submanifold N of a semi-Riemannian manifold M is a semi-Riemannian submanifold if for each $x \in N$, the restriction of $< \cdot, \cdot >_{(x)}$ to T_xN is nondegenerate. There are several important submanifolds of $\mathbb{R}^{n+1,1}$. One such is the Lorentzian submanifold

$$\Lambda^n \; = \; \{(v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1,1} : \sum_{i=1}^{n} v_i^2 \, - \, v_{n+1}^2 \, = \, 1\},$$

which is called de Sitter space (see Fig. 3). A second important one is *hyperbolic space* $\mathbb{H}^n$ defined by

$$\mathbb{H}^n \; = \; \{(v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1,1} : \sum_{i=1}^{n} v_i^2 \, - \, v_{n+1}^2 \, = \, 1 \;\text{ and }\; v_{n+2} > 0\}.$$

By contrast the restriction of $< \cdot, \cdot >_L$ to $\mathbb{H}^n$ is a Riemannian metric of constant negative curvature -1. There is natural duality between codimension 1 submanifolds of $\mathbb{H}^n$ obtained as the intersection of $\mathbb{H}^n$ with a "time-like"hyperplane Π through 0 (containing a "time-like"vector z with $< z, z >_L < 0$) paired with the points $\pm z' \in \Lambda^n$ given where z' lies on a line through the origin which is the Lorentzian orthogonal complement to Π.

Many of the results which hold for Riemannian manifolds also hold for a Semi-Riemannian manifold M.

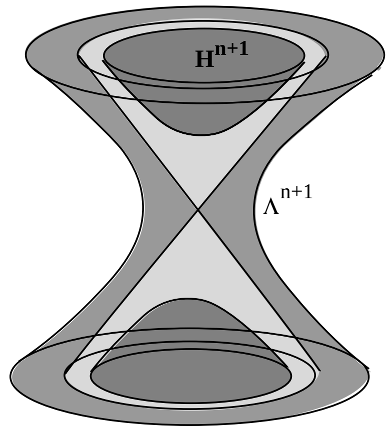

FIGURE 3. In Minkowski space $\mathbb{R}^{n+2,1}$, there is the Lorentzian hypersurface Λ^{n+1} and the model for hyperbolic space $\mathbb{H}^{n+1}$. Also shown is the "light cone".

2.1 (Basic properties of Semi-Riemannian Manifolds (see [ON]).

For a Semi-Riemannian manifold M, there are the following properties analogous to those for Riemannian manifolds:

1. *Smooth Curves on M have lengths defined using $| < \cdot, \cdot > |$.*
2. *There is a unique connection which satisfies the usual properties of a Riemannian Levi-Civita connection.*
3. *Geodesics are defined locally from any point $x \in M$ and with any initial velocity $v \in T_x M$. They are critical curves for the length functional, and they have constant speed.*
4. *If N is a semi-Riemannian submanifold of M, then a constant speed curve $\gamma(t)$ in N is a geodesic in N if the acceleration $\gamma''(t)$ is normal to N (with respect to the semi-Riemannian metric) at all points of $\gamma(t)$.*
5. *Any point $x \in M$ has a "convex neighborhood"W, which has the property that any two points in W are joined by a unique geodesic in the neighborhood.*
6. *If $\gamma(t)$ is a geodesic joining $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$ and x_0 and x_1 are not conjugate along $\gamma(t)$, then given a neighborhood W of $\gamma(t)$, there are neighborhoods of W_0 of x_0 and W_1 of x_1 so that if $x'_0 \in W_0$, and $x'_1 \in W_1$, there is a unique geodesic in the neighborhood W from x'_0 to x'_1.*

As an example, it is straightforward to verify that for any $z \in \Lambda$, the vector z is orthogonal to Λ at the point z. Suppose P is a plane in $\mathbb{R}^{n+1,1}$ containing the origin. Let $\gamma(t)$ be a constant Lorentzian speed parametrization of the curve obtained by intersecting P with Λ. Then, by a standard argument similar to that for the case of a Euclidean sphere, $\gamma(t)$ is a geodesic. All geodesics of Λ are obtained in this way. It follows that the submanifolds of Λ obtained by intersecting Λ with a linear subspace is a totally geodesic submanifold of Λ.

3. Dual Varieties and Singular Lorentzian Manifolds

Given a smooth hypersurface $M \in \mathbb{R}^n$, we define a natural map from M to Λ^{n+1}. First, we let S^{n+1} denote the unit sphere in $\mathbb{R}^{n+1}$ centered at the origin, and we let $\mathbf{e}_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Then, stereographic projection defines a map $p : S^{n+1} \backslash \{\mathbf{e}_{n+1}\} \to \mathbb{R}^n$ sending y to the point where the line from $\mathbf{e}_{n+1}$ to y intersects $\mathbb{R}^n$. Given a hyperplane Π in $\mathbb{R}^n$, it together with $\mathbf{e}_{n+1}$ spans a hyperplane Π' in $R^{n+1} \times \{\mathbf{e}_{n+2}\}$. The intersection of this plane with S^{n+1} is an n-sphere. Identifying R^{n+1} with the hyperplane in $\mathbb{R}^{n+2\,1}$ defined by $x_{n+2} = 1$. Then, Π' together with 0 spans a hyperplane Π'' in $\mathbb{R}^{n+2\,1}$. This hyperplane is time-like because Π'' intersects $R^{n+1} \times \{\mathbf{e}_{n+2}\}$ in a hyperplane Π' which intersects the unit sphere in $R^{n+1} \times \{\mathbf{e}_{n+2}\}$ in a sphere, hence it intersects the interior diisk. Then, the duality associates a pair of points z and $-z$ in Λ^{n+1} which lie on a common line through the origin.

In order to obtain a single valued map, there are two possibilities: Either we consider the induce map to $\tilde{\Lambda}^{n+1} = \Lambda^{n+1}/\sim$, where $\sim$ identifies each pair of points z and $-z$ of Λ^{n+1}; or we need on M a smooth normal unit vector field $\mathbf{n}$ orienting M. Given the normal vector field $\mathbf{n}$, it defines a distinguished side of T_xM. If this is Π, then we obtain a distinguished side for Π' and then Π'', which singles out one of the two points in Λ^{n+1} on the distinguished side. We shall refer to this second case as the oriented case.

We shall use both versions of the maps. In fact, the image lies in the submanifold $\mathcal{R}$ of Λ^{n+1} defined by

$$\mathcal{R} \quad = \quad \{(\mathbf{n}, c\boldsymbol{\epsilon}) : \mathbf{n} \in S^{n-1}, c \in \mathbb{R}\}$$

which we can view as a submanifold $\mathcal{R} \subset \Lambda^{n+1}$; or in the general case it lies in $\tilde{\mathcal{R}}$. We denote the general form of the map by $\tilde{\mathcal{L}} : M \to \tilde{\mathcal{R}}$, and the oriented form by $\mathcal{L} : M \to \mathcal{R}$.

We can give a coordinate definitions for the maps. If T_xM is defined by $\mathbf{n} \cdot \mathbf{x} = c$, where $\mathbf{x} = (x_1, \ldots, x_n)$. Then, Π' contains T_xM and $\mathbf{e}_{n+1}$ and so is defined by $\mathbf{n} \cdot \mathbf{x} + cx_{n+1} = c$. Then, Π'' contains $\Pi' \times \{\mathbf{e}_{n+2}\}$ and the origin so it is defined by $\mathbf{n} \cdot \mathbf{x} + cx_{n+1} - cx_{n+2} = 0$. Thus, the Lorentzian orthogonal line is spanned by $(\mathbf{n}, c, c)$, which we write in abbreviated form as $(\mathbf{n}, c\boldsymbol{\epsilon})$ with $\boldsymbol{\epsilon} = (1, 1)$. Hence, the map $\mathcal{L} : M \to \Lambda^{n+1}$ sends x to $(\mathbf{n}, c\boldsymbol{\epsilon})$, and the general case sends it to the equivalence class in $\tilde{\mathcal{R}}$ determined by $(\mathbf{n}, c\boldsymbol{\epsilon})$. We shall be concerned with a subspace of Λ^{n+1} where this duality corresponds to hypersurfaces of $\mathbb{R}^n$. The general correspondence is used in [OH] to parametrize $(n-1)$-dimensional spheres in $\mathbb{R}^n$.

Definition 3.1. Given a smooth hypersurface $M \in \mathbb{R}^n$, with a smooth normal vector field $\mathbf{n}$ on M, the *(oriented) Lorentz map* is the natural map $\mathcal{L} : M \to \mathcal{R}$ defined by $\mathcal{L}(x) = (\mathbf{n}, c\boldsymbol{\epsilon})$, where T_xM is defined by $\mathbf{n} \cdot \mathbf{x} = c$. In the general case, we choose a local normal vector field and then $\tilde{\mathcal{L}}(x)$ is the equivalence class of $(\mathbf{n}, c\boldsymbol{\epsilon})$ in $\tilde{\mathcal{R}}$.

In the following we shall generally concentrate on the oriented case and the map $\mathcal{L}$, with the general case just involving considering the map to equivalence classes. There are two questions concerning $\mathcal{L}$. One is when $\mathcal{L}$ is nonsingular, and at singular points what can we say about the local properties of $\mathcal{L}$ when M is generic. The second question is how we may construct the inverse of $\mathcal{L}$ when it is a local embedding (or immersion).

Relation with the Dual Variety. Suppose that $M \subset \mathbb{R}^n$ is a smooth hypersurface. There is a natural way to associate a corresponding "dual variety" $M^\vee$ in the dual projective space $\mathbb{R}P^{n\,\vee}$ (which consists of lines through the origin in the dual space $\mathbb{R}^{n+1\,*}$). Given a hyperplane $\Pi \subset \mathbb{R}^n$, it is defined by an equation $\sum_{i=1}^n a_i x_i = b$. We associate the linear form $\alpha : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $\alpha(x_1, \dots, x_{n+1}) = \sum_{i=1}^n a_i x_i - b x_{n+1}$. As the equation for Π is only well defined up to multiplication by a constant, so is α, which defines a unique line in $\mathbb{R}^{n+1\,*}$. This then defines a *dual mapping* $\delta : M \to \mathbb{R}P^{n\,\vee}$, sending $x \in M$ to the dual of $T_x M$.

In the context of algebraic geometry in the complex case, this map actually extends to a dual map for a smooth codimension 1 algebraic subvariety $M \subset \mathbb{C}P^n$, and then the image $M^\vee = \delta(M)$ is again a codimension 1 algebraic subvariety of $\mathbb{C}P^{n\,\vee}$. There is an inverse dual map $\delta^\vee$ for smooth codimension 1 algebraic subvarieties of $\mathbb{C}P^{n\,\vee}$ to $\mathbb{C}P^n$ defined again using the tangent spaces. Hence, $\delta^\vee : M^\vee \to \mathbb{C}P^n$. It is only defined on smooth points of $M^\vee$ (which may have singularities); however it extends to the singular points of $M^\vee$ and its image is the original M.

In our situation, we are working over the reals and moreover M will not be defined by algebraically. Hence, we need to determine what properties both δ and $M^\vee$ have. We also will explain the relation with the Lorentz map.

Legendrian Projections. Given M, we let $P(\mathbb{R}^{n+1\,*})$ denote the projective bundle $\mathbb{R}^n \times \mathbb{R}P^{n\,\vee}$, where as earlier $\mathbb{R}P^{n\,\vee}$ denotes the dual projective space. Then, we have an embedding $i : M \to P(\mathbb{R}^{n+1\,*})$, where $i(x) = (x, < \alpha_x >)$, with α_x the linear form associated to $T_x M$ as above. We let $\tilde{M} = i(M)$. There is a projection map $\pi : P(\mathbb{R}^{n+1\,*}) \to \mathbb{R}P^{n\,\vee}$. Then, by results in Arnol'd, π is a Legendrian projection, and for generic M, $\tilde{M}$ is a generic Legendrian submanifold of $P(\mathbb{R}^{n+1\,*})$ and the restriction $\pi|\tilde{M} : \tilde{M} \to \mathbb{R}P^{n\,\vee}$ is a generic Legendrian projection. This composition $\pi|\tilde{M} \circ i$ is exactly δ. Hence, the properties of δ are exactly those of the Legendrian projection. In particular, the singularities of $M^\vee = \pi(\tilde{M})$ are generic Legendrian singularities, which are the singularities appearing in discriminants of stable mappings, see [A1] or [AGV, Vol 2].

In the case of surfaces in $\mathbb{R}^3$, these are: cuspidal edge, a swallowtail, transverse intersections of two or three smooth surfaces, and the transverse intersection of a smooth surface with a cuspidal edge (as shown in Fig. 4). The characterization of these singularities implies that as we approach a singular point from one of the connected components, then there is a unique limiting tangent plane, and in the case of the cuspidal edge or swallowtail, the limiting tangent plane is the same for each component. Hence, for generic smooth hypersurfaces $M \subset \mathbb{R}^n$, the inverse dual map $\delta^\vee$ extends to all of $M^\vee$, and again will have image M.

Finally, we remark about the relation between the dual variety $M^\vee$ and the image $M_{\mathcal{L}} = \mathcal{L}(M)$ (or $M_{\tilde{\mathcal{L}}} = \tilde{\mathcal{L}}(M)$). To do so, we introduce a mapping involving $\mathbb{R}P^{n\,\vee}$ and $\tilde{\mathcal{R}}$. In $\mathbb{R}P^{n\,\vee}$, there is the distinguished point $\infty =< (0, \dots, 0, 1) >$. On

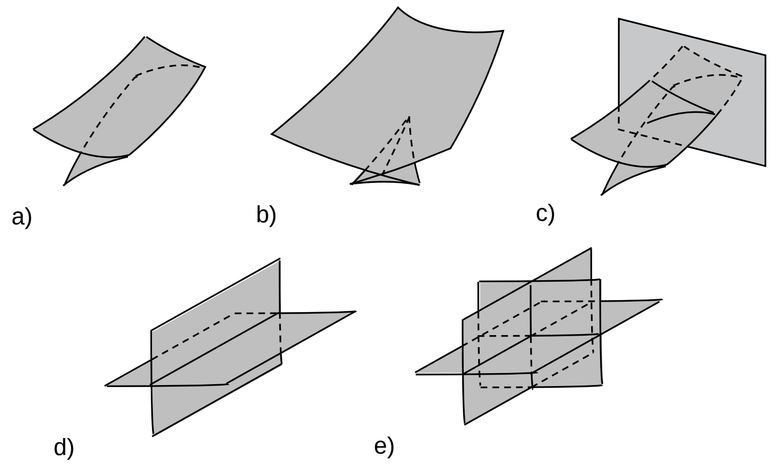

FIGURE 4. Generic Singularities for Legendrian projections of Legendrian surfaces: a) cuspidal edge, b) swallowtail, c) transverse intersection of cuspidal edge and smooth surface, d) transverse intersection of two smooth surfaces, and e) transverse intersection of three smooth surfaces.

$\mathbb{R}P^{n\vee}\backslash\{\infty\}$, we may take a point $< (y_1, \dots, y_n, y_{n+1}) >$, and normalize it by

$$(y'_1, \dots, y'_n, y'_{n+1}) = c \cdot (y_1, \dots, y_n, y_{n+1}), \quad \text{where } c = (\sum_{i=1}^{n} y_i^2)^{-\frac{1}{2}}.$$

Then, $\mathbf{n}_y = (y'_1, \dots, y'_n)$ is a unit vector. We then define a map $\nu : \mathbb{R}P^{n\vee}\backslash\{\infty\} \to \tilde{\mathcal{R}}$ sending $< (y_1, \dots, y_n, y_{n+1}) >$ to $(\mathbf{n}_y, y'_{n+1}\boldsymbol{\epsilon})$. This is only well defined up to multiplication by -1, which is why we must take the equivalence class in the pair of points. If we are on a region of $\mathbb{R}P^{n\vee}\backslash\{\infty\}$ where we can smoothly choose a direction for each line corresponding to a point in $\mathbb{R}P^{n\vee}$, then as for the case of the Lorentzian mapping, we can give a well-defined map to $\mathcal{R}$. This will be so when we consider $M^\vee$ for the oriented case. In such a situation, when the smooth hypersurface M has a smooth unit normal vector field $\mathbf{n}$, it provides a positive direction in the line of linear forms vanishing on T_xM.

Then, we have the following relations.

Lemma 3.2. *The smooth mapping* $\tilde{\nu} : \mathbb{R}P^{n\vee}\backslash\{\infty\} \to \tilde{\mathcal{R}}$ *is a diffeomorphism.*

Second, there is the relation between the duality map δ and the Lorentz map $\tilde{\mathcal{L}}$ (or $\mathcal{L}$).

Lemma 3.3. *If* $M \subset \mathbb{R}^n$ *is a smooth hypersurface, then the diagram (3.1) commutes, i.e.* $\tilde{\nu} \circ \delta = \tilde{\mathcal{L}}$. *If, in addition,* M *has a smooth unit normal vector field* $\mathbf{n}$, *then there is the oriented version of diagram (3.1),* $\nu \circ \delta = \mathcal{L}$.

$$\begin{array}{ccc} M & \xrightarrow{\delta} & \mathbb{R}P^{n\vee} \\ & \underset{\tilde{\mathcal{L}}}{\searrow} & \downarrow \tilde{\nu} \\ & & \tilde{\mathcal{R}} \end{array} \tag{3.1}$$

Lorentzian As a consequence of these Lemmas and our earlier discussion about

the singularities of $M^\vee$, we conclude that $M_{\tilde{\mathcal{L}}}$ (or $M_{\mathcal{L}}$) have the same singularities. Thus, we may suppose they are generic Legendrian singularities. Although we have by Lemma 3.2 that $\mathbb{R}P^{n\,\vee}\backslash\{\infty\}$ is diffeomorphic to $\tilde{\mathcal{R}}$, the first space has a natural Riemannian structure while on $\tilde{\mathcal{R}}$ we have a Lorentzian metric.

Proof of Lemma 3.2. There is a natural inverse to $\tilde{\nu}$ defined as follows: If $z = (\mathbf{n}, c\boldsymbol{\epsilon})$ and $\mathbf{n} = (a_1, \dots, a_n)$, then we map z to $< (a_1, \dots, a_n, -c) >$. We note that replacing z by $-z$ does not change the line $< (a_1, \dots, a_n, -c) >$. This gives a well-defined smooth map $\tilde{\mathcal{R}} \to \mathbb{R}P^{n\,\vee}\backslash\{\infty\}$ which is easily checked to be the inverse of $\tilde{\nu}$. □

Proof of Lemma 3.3. If T_xM is defined by $\mathbf{n} \cdot \mathbf{x} = c$ with $\mathbf{n} = (a_1, \dots, a_n)$, then $\delta(x) =< (a_1, \dots, a_n, -c) >$. Then, as $\|\mathbf{n}\| = 1$, $\tilde{\nu}(< (a_1, \dots, a_n, -c) >) = (a_1, \dots, a_n, c, c) = (\mathbf{n}, c\boldsymbol{\epsilon})$, which is exactly $\mathcal{L}(x)$. □

Inverses of the Dual Variety and Lorentzian Mappings. We consider how to invert both δ and $\tilde{\mathcal{L}}$. We earlier remarked that in the complex algebraic setting, the inverse to δ is again a dual map $\delta^\vee$. As $\tilde{\nu}$ is a diffeomorphism, and diagram 3.1 commutes, inverting δ is equivalent to inverting $\tilde{\mathcal{L}}$. Also, constructing an inverse is a local problem, so we may as well consider the oriented case.

Proposition 3.4. *Let $M \subset \mathbb{R}^n$ be a generic smooth hypersurface with a smooth unit normal vector field $\mathbf{n}$. Suppose that the image $M_{\mathcal{L}}$ under $\mathcal{L}$ is a smooth submanifold of $\mathcal{R}$. Then, M is obtained as the envelope of the collection of hyperplanes defined by $\mathbf{n} \cdot \mathbf{x} = c$ for $\mathcal{L}(x) = (\mathbf{n}, c\boldsymbol{\epsilon})$.*

Proof of Proposition 3.4. We consider an $(n-1)$-dimensional submanifold of $\mathcal{R}$ parametrized by $u \in U$ given by $(\mathbf{n}(u), c(u)\boldsymbol{\epsilon})$. The collection of hyperplanes are given by Π_u defined by $F(\mathbf{x}, u) = \mathbf{n}(u) \cdot \mathbf{x} - c(u) = 0$. Then, the envelope is defined by the collection of equations $F_{u_i} = 0, i = 1, \dots, n-1$ and $F = 0$. This is the system of linear equations

$$(3.2) \qquad i)\ \mathbf{n}(u) \cdot \mathbf{x} = c(u) \qquad \text{and } ii)\ \mathbf{n}_{u_i}(u) \cdot \mathbf{x} = c_{u_i}(u),\ i = 1, \dots, n-1$$

A sufficient condition that there exist for a given u a unique solution to the system of linear equations in $\mathbf{x}$ is that the vectors $\mathbf{n}, \mathbf{n}_{u_1}, \dots, \mathbf{n}_{u_{n-1}}$ are linearly independent. Since $\mathbf{n}_{u_i} = -S(\dfrac{\partial}{\partial u_i})$, for S the shape operator for M, linear independence is equivalent to S not having any 0-eigenvalues. Thus, $\mathbf{x}$ is not a parabolic point of M. For generic M, the set of parabolic points is a statified set of codimension 1 in M. Thus, off the image of this set, there is a unique point in the envelope.

Also, if we differentiate equation (3.2)-i) with respect to u_i we obtain

$$(3.3) \qquad \mathbf{n}_{u_i}(u) \cdot \mathbf{x} + \mathbf{n}(u) \cdot \mathbf{x}_{u_i} \;=\; c_{u_i}(u)$$

Combining this with (3.2)-ii), we obtain

$$(3.4) \qquad \mathbf{n}(u) \cdot \mathbf{x}_{u_i} \;=\; 0,$$

and conversely, (3.4) for $i = 1, \dots, n-1$ and (3.3) imply (3.2)-ii). Thus, if we choose a local parametrization of M given by $\mathbf{x}(u)$, then as $\mathbf{x}(u)$ is a point in its tangent space, it satisfies (3.2)-i), and hence (3.3), and also $\mathbf{n}$ being a normal vector field implies that (3.4) is satisfied for all i. Thus, (3.2)-ii) is satisfied. Hence, M is part of the envelope. Also, for generic points of M, by the implicit function theorem, the set of solutions of (3.2) is locally a submanifold of dimension $n-1$. Hence, in a neighborhood of these generic points of M, the envelope is exactly M. Hence, the closure of this set is all of M and still consists of solutions of (3.2). Thus, we recover M.

Second, to see that the equations (3.2) describe the inverse of the dual mapping, we note by Lemmas 3.2 and 3.2 that $\tilde{\nu}$ is a diffeomorphism, $\delta^{-1} = \tilde{\mathcal{L}}^{-1} \circ \tilde{\nu}$, and the preceding argument gives the local inverse to $\tilde{\mathcal{L}}$. □

4. Lorentzian Geodesic Flow on Λ^{n+1}

We give the general formula for the geodesic flow between $z_0 = (\mathbf{n}_0, d_0\boldsymbol{\epsilon})$ and $z_1 = (\mathbf{n}_1, d_1\boldsymbol{\epsilon})$.

Several Auxiliary Functions.

To do so we introduce several auxiliary functions. We first define the function $\lambda(x, \theta)$ by

$$\lambda(x,\theta) \;=\; \begin{cases} \frac{\sin(x\theta)}{\sin(\theta)} & \theta \neq 0 \\ x & \theta = 0 \end{cases} \tag{4.1}$$

Then, $\sin(z)$ is a holomorphic function of z, and the quotient $\frac{\sin(x\theta)}{\sin(\theta)}$ has removable singularities along $\theta = 0$ with value x. Hence, $\lambda(z, \theta)$ is a holomorphic function of (z, θ) on $\mathbb{C} \times (-\pi, \pi)$, and so analytic on $\mathbb{R} \times (-\pi, \pi)$. Also, directly computing the derivative we obtain

$$\frac{\partial \lambda((x,\theta)}{\partial x} \;=\; \begin{cases} \cos(x\theta)\cdot\frac{\theta}{\sin\theta} & \theta \neq 0 \\ 1 & \theta = 0 \end{cases} \tag{4.2}$$

Remark . In fact, we can recognize $\lambda(n, \theta)$ for integer values n as the characters for the irreducible representations of $SU(2)$ restricted to the maximal torus.

We also introduce a second function for later use in §7. For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we define

$$\mu(x,\theta) \;=\; \frac{\cos(x\theta)}{\cos(\theta)}.$$

Then, there is the following relation

$$\lambda(x,\theta) + \lambda(1-x,\theta) \;=\; \mu(1-2x, \frac{\theta}{2}) \tag{4.3}$$

This follows by using the basic trigonometric formulas $\sin(x) + \sin(y) = 2\cos(\frac{1}{2}(x+y))\sin(\frac{1}{2}(x-y))$ and $\sin\theta = 2\sin(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta)$. There are additional relations between these two functions that follow from other basic trigonometric identies.

Geodesic Curves in Λ^{n+1} joining points in $\mathcal{R}$.

We may express the geodesic curve between z_0 and z_1 in Λ^{n+1} using $\lambda(x, \theta)$. We let $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ be defined by $\cos\theta = \mathbf{n}_0 \cdot \mathbf{n}_1$.

Proposition 4.1. *The geodesic curve $\gamma(t)$ in Λ from $\gamma(0) = z_0$ and $\gamma(1) = z_1$ for the Lorentzian metric on Λ is given by*

$$\gamma(t) \quad = \quad \lambda(t, \theta)\, z_1 \,+\, \lambda(1-t, \theta)\, z_0 \qquad \text{for } 0 \leq t \leq 1 \tag{4.4}$$

This curve lies in $\mathcal{R}$ for $0 \leq t \leq 1$.

Remark 4.2 (Invariance of Lorentzian Geodesic Flow)**.** The Geodesic flow given in Proposition 4.1 is invariant under the group of rigid transformations and scalar multiplications. By this we mean the following. Suppose $z_i = (\mathbf{n}_i, c_i) \in \mathcal{R}$, i = 1, 2. Let Π_i be the hyperplane determined by z_i. Let ψ be a composition of scalar multiplication by b followed by a rigid transformation so $\psi(\mathbf{x}) = b\, A(\mathbf{x}) + \mathbf{p}$, with A an orthogonal transformation. Then, $\Pi'_i = \psi(\Pi_i)$ is defined by

$$\tilde{\psi}(z_i) = \tilde{\psi}(\mathbf{n}_i, c_i) = (A(\mathbf{n}_i), bc_i + \mathbf{n}_i \cdot \mathbf{p}).$$

If $\gamma(t) = (\mathbf{n}_t, c_t)$ is the Lorentzian geodesic flow between z_0 and z_1, then $\tilde{\psi}(\gamma(t)$ is the Lorentzian geodesic flow between $\tilde{\psi}(z_0)$ and $\tilde{\psi}(z_1)$. See §7.

We can expand the expression for $\gamma(t)$ and obtain the family of hyperplanes Π_t in $\mathbb{R}^n$. Expanding (4.4) we obtain

$$\begin{aligned} n_t \quad &= \quad \lambda(t, \theta)\, \mathbf{n}_1 \,+\, \lambda(1-t, \theta)\, \mathbf{n}_0 \quad \text{and} \\ c_t \quad &= \quad \lambda(t, \theta)\, c_1 \,+\, \lambda(1-t, \theta)\, c_0 \end{aligned} \tag{4.5}$$

Then the family Π_t is given by

$$\Pi_t \quad = \quad \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{n}_t = c_t\} \tag{4.6}$$

We can also compute the initial velocity for the geodesic in (4.4).

Corollary 4.3. *The initial velocity of the geodesic (4.4) with $\theta \neq 0$ is given by*

$$\gamma'(0) \quad = \quad \frac{\theta}{\sin\theta} \cdot (\mathrm{proj}_{\mathbf{n}_0}(\mathbf{n}_1), (c_1 - \cos\theta\, c_0)\boldsymbol{\epsilon}) \tag{4.7}$$

where $\mathrm{proj}_{\mathbf{n}_0}$ denotes projection along $\mathbf{n}_0$ onto the line spanned by $\mathbf{w}$. If $\theta = 0$, then $\mathbf{n}_0 = \mathbf{n}_1$ and the velocity is $(0, (c_1 - c_0)\boldsymbol{\epsilon})$ (with Lorentzian speed 0).

Remark . Note that

$$\|(\mathrm{proj}_{\mathbf{n}_0}(\mathbf{n}_1), (c_1 - \cos\theta\, c_0)\boldsymbol{\epsilon})\|_L \;=\; \|\mathrm{proj}_{\mathbf{n}_0}(\mathbf{n}_1)\|$$

which equals $\sin\theta$. We conclude that the Lorentzian magnitude of $\gamma'(0)$ is θ. Since geodesics have constant speed, the geodesic will travel a distance $|\theta|$. Hence, $|\theta|$ is the Lorentzian distance between z_0 and z_1.

Proof of Proposition 4.1. Let P be the plane in $\mathbb{R}^{n+1,1}$ which contains 0, z_0 and z_1. The geodesic curve between z_0 and z_1 is obtained as a constant Lorentzian speed parametrization of the curve obtained by intersecting P with Λ. We choose a unit vector $\mathbf{w} \in \Pi$ such that $\mathbf{n}_1$ is in the plane through the origin spanned by $\mathbf{n}_0$ and $\mathbf{w}$. Let θ be the angle between $\mathbf{n}_0$ and $\mathbf{n}_1$ so $\cos\theta = \mathbf{n}_0 \cdot \mathbf{n}_1$. Then, $\mathbf{n}_1 - (\mathbf{n}_1 \cdot \mathbf{n}_0)\, \mathbf{n}_0$ is the projection of $\mathbf{n}_1$ along $\mathbf{n}_0$ onto the line spanned by $\mathbf{w}$. It equals $\mathbf{n}_1 - \cos\theta\, \mathbf{n}_0 = \sin\theta\, \mathbf{w}$.

Then, a tangent vector to $\Lambda^{n+1} \cap P$ at the point z_0 is given by

$$(\mathbf{n}_1 - \cos\theta\, \mathbf{n}_0, (c_1 - \cos\theta\, c_0)\boldsymbol{\epsilon}) = (\sin\theta\, \mathbf{w}, (c_1 - \cos\theta\, c_0)\boldsymbol{\epsilon}) \tag{4.8}$$

Then, we seek a Lorentzian geodesic $\gamma(t)$ in the plane P beginning at $(\mathbf{n}_0, c_0\boldsymbol{\epsilon})$ with initial velocity in the direction $(\sin\theta\,\mathbf{w}, (c_1 - \cos\theta\, c_0)\boldsymbol{\epsilon})$. Consider the curve

$$\gamma(t) \;=\; (\cos(t\theta)\mathbf{n}_0 \,+\, \sin(t\theta)\mathbf{w}, (\cos(t\theta)c_0 \,+\, \frac{\sin(t\theta)}{\sin(\theta)}(c_1 - \cos\theta\, c_0))\boldsymbol{\epsilon}) \tag{4.9}$$

First, note that $\gamma(0) = z_0$, and $\gamma(1) = z_1$. Also, this curve lies in the plane spanned by z_0 and (4.8). Also,

$$\|\gamma(t)\|_L \;=\; \|\cos(t\theta)\mathbf{n}_0 \,+\, \sin(t\theta)\mathbf{w}\| \;=\; 1$$

as $\mathbf{n}_0$ and $\mathbf{w}$ are orthogonal unit vectors. Hence, $\gamma(t)$ is a curve parametrizing $\Lambda^{n+1} \cap P$. It remains to show that γ'' is Lorentzian orthogonal to Λ^{n+1} to establish that it is a Lorentzian geodesic from z_0 to z_1. A computation shows

$$\gamma''(t) \;=\; -\theta^2(\cos(t\theta)\mathbf{n}_0 \,+\, \sin(t\theta)\mathbf{w}, \frac{\sin(t\theta)}{\sin(\theta)}(c_1 - \cos\theta\, c_0)\boldsymbol{\epsilon})$$

which is $-\theta^2\gamma(t)$, and hence Lorentzian orthogonal to Λ^{n+1}.

Because of the fraction $\frac{\sin(t\theta)}{\sin(\theta)}$, we have to note that when $\theta = 0$, then $\mathbf{n}_0 = \mathbf{n}_1$ and $\gamma(t)$ takes the simplified form

$$\gamma(t) \;=\; (\mathbf{n}_0, c_0 \,+\, t(c_1 - c_0))\boldsymbol{\epsilon})$$

which is still a Lorentzian geodesic between z_0 to z_1.

Lastly, we must show that this agrees with (4.4). First, consider the case where $\theta \neq 0$.

$$\mathbf{w} = \frac{1}{\sin\theta}\,(\mathbf{n}_1 - \cos\theta\,\mathbf{n}_0)$$

Substituting this into the first term of the RHS of (4.9), we obtain

$$\frac{1}{\sin\theta}(\sin\theta\,\cos(t\theta) - \cos\theta\,\sin(t\theta))\,\mathbf{n}_0 \,+\, \frac{sin(t\theta)}{\sin\theta}\,\mathbf{n}_1$$

which by the formula for the sine of the difference of two angles equals

$$\frac{\sin((1-t)\theta)}{\sin\theta}\mathbf{n}_0 \,+\, \frac{sin(t\theta)}{\sin\theta}\mathbf{n}_1$$

Analogously, we can compute the second term in the RHS of (4.9), to be

$$\frac{\sin((1-t)\theta)}{\sin\theta}c_0 \,+\, \frac{sin(t\theta)}{\sin\theta}c_1$$

This gives (4.4) when $\theta \neq 0$. When $\theta = 0$, $\mathbf{n}_0 = \mathbf{n}_1$ and the derivation of (4.4) from (4.9) for $\theta = 0$ is easier. □

5. **Sufficient Condition for Smoothness of Envelopes**

To describe the induced "geodesic flow"between hypersurfaces M_0 and M_1 in $\mathbb{R}^n$, we will use the Lorentzian geodesic flow in $\mathcal{R}$ and then find the corresponding flow by applying an inverse to $\mathcal{L}$. We begin by constructing the inverse for a $(n-1)$-dimensional manifold in $\mathcal{R}$ parametrized by $(\mathbf{n}(u), c(u)\boldsymbol{\epsilon})$, where $u = (u_1, \ldots, u_{n-1})$. We determine when the associated family of hyperplanes $\Pi_u = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n}(u)\cdot\mathbf{x} = c(u)\}$. has envelope a smooth hypersurface in $\mathbb{R}^n$.

We introduce a family of vectors in $\mathbb{R}^{n+1}$ given by $\tilde{\mathbf{n}}(u) = (\mathbf{n}(u), -c(u))$. We also denote $\dfrac{\partial\tilde{\mathbf{n}}}{\partial u_i}$ by $\tilde{\mathbf{n}}_{u_i}$. Next we consider the n-fold cross product in $\mathbb{R}^{n+1}$, denoted by $v_1 \times v_2 \times \cdots \times v_n$, which is the vector in $\mathbb{R}^{n+1}$ whose i-th coordinate is $(-1)^{i+1}$

times the $n \times n$ determinant obtained from the entries of $v_1, \dots, v_n$ by removing the i-th entries of each v_j. Then, for any other vector v,

$$v \cdot (v_1 \times v_2 \times \cdots \times v_n) \;=\; \det(v, v_1, \dots, v_n)$$

We let

$$\tilde{\mathbf{h}} \;=\; \tilde{\mathbf{n}} \times \tilde{\mathbf{n}}_{u_1} \times \cdots \times \tilde{\mathbf{n}}_{u_{n-1}}$$

We let $H(\tilde{\mathbf{n}})$ denote the $(n-1) \times (n-1)$ matrix of vectors $\tilde{\mathbf{n}}_{u_i\, u_j}$. Then we can form $H(\tilde{\mathbf{n}}) \cdot \tilde{\mathbf{h}}$ to be the $(n-1) \times (n-1)$ matrix with entries $\tilde{\mathbf{n}}_{u_i\, u_j} \cdot \tilde{\mathbf{h}}$. Then, there is the following determination of the properties of the envelope of $\{\Pi_u\}$.

Proposition 5.1. *Suppose we have an $(n-1)$-dimensional manifold in $\mathcal{R}$ parametrized by $(\mathbf{n}(u), c(u)\boldsymbol{\epsilon})$, where $u = (u_1, \dots, u_{n-1})$. We let $\{\Pi_u\}$ denote the associated family of hyperplanes. Then, the envelope of $\{\Pi_u\}$ has the following properties.*

i) *There is a unique point $\mathbf{x}_0$ on the envelope corresponding to u_0 provided $\mathbf{n}(u_0), \mathbf{n}_{u_1}(u_0), \dots, \mathbf{n}_{u_{n-1}}(u_0)$ are linearly independent.*
ii) *Provided i) holds, the envelope is smooth at $\mathbf{x}_0$ provided $H(\tilde{\mathbf{n}}) \cdot \tilde{\mathbf{h}}$ is nonsingular for $u = u_0$.*
iii) *Provided ii) holds, the normal to the surface at $\mathbf{x}_0$ is $\mathbf{n}(u_0)$ and Π_{u_0} is the tangent plane at $\mathbf{x}_0$.*

Proof of Proposition 5.1. We use the line of reasoning for Proposition 3.4. the condition that a point $\mathbf{x}_0$ belong to the envelope of $\{\Pi_u\}$ is that it satisfy the system of equations (3.2). A sufficient condition that these equations have a unique solution for $u = u_0$ is exactly that $\mathbf{n}(u_0), \mathbf{n}_{u_1}(u_0), \dots, \mathbf{n}_{u_{n-1}}(u_0)$ are linearly independent.

Furthermore, if this is true at u_0 then it is true in a neighborhood of u_0. Thus, we have a unique smooth mapping $\mathbf{x}(u)$ from a neighborhood of u_0 to $\mathbb{R}^n$. By the argument used to deduce (3.4), we also conclude

$$\mathbf{n}(u) \cdot \mathbf{x}_{u_i} \;=\; 0, \qquad i = 1, \dots, n-1 \tag{5.1}$$

Hence, if $\mathbf{x}(u)$ is nonsingular at u_0, then $\mathbf{n}(u_0)$ is the normal vector to the envelope hypersurface at $\mathbf{x}_0$, so the tangent plane is Π_{u_0}. Thus iii) is true.

It remains to establish the criterion for smoothness in ii). As earlier mentioned the envelope in the neighborhood of a point $\mathbf{x}_0$ is the discriminant of the projection of $V = \{(\mathbf{x}, u) : F(\mathbf{x}, u) = \mathbf{n}(u) \cdot \mathbf{x} - c(u) = 0\}$ to $\mathbb{R}^n$. It is a standard classical result that at a point $(\mathbf{x}_0, u_0) \in V$, which projects to an envelope point $\mathbf{x}_0$, the envelope is smooth at $\mathbf{x}_0$ provided $(\mathbf{x}_0, u_0)$ is a regular point of F (so V is smooth in a neighborhood of $(\mathbf{x}_0, u_0)$) and the partial Hessian $(\dfrac{\partial^2 F}{\partial u_i\, u_j}(\mathbf{x}_0, u_0))$ is nonsingular. For our particular F this Hessian becomes $H(\mathbf{n}) \cdot \mathbf{x}_0 - H(c)$, where $H(\mathbf{n})$ is the $n \times n$ matrix $(\mathbf{n}_{u_i\, u_j})$, and $H(\mathbf{n}) \cdot \mathbf{x}_0$ is the $(n-1) \times (n-1)$ matrix whose entries are $\mathbf{n}_{u_i\, u_j} \cdot \mathbf{x}_0$.

Now $\mathbf{x}_0$ is the unique solution of the system of linear equations (3.2). This solution is given by Cramer's rule. Let $N(u_0)$ denote the $n \times n$ matrix with columns $\mathbf{n}(u_0), , \mathbf{n}_{u_1(u_0)}, \dots, \mathbf{n}_{u_{n-1}(u_0)}$. Then, by Cramer's rule, if we multiply $\mathbf{x}_0$ by $\det(N(u_0))$ we obtain $(-1)^n \tilde{\mathbf{h}}$. Thus, multiplying $H(\mathbf{n}) \cdot \mathbf{x}_0 - H(c)$ by $\det(N(u_0))$ yields $(-1)^n (H(\mathbf{n}), -H(c)) \cdot \tilde{h}$ which is exactly $(-1)^n H(\tilde{\mathbf{n}}) \cdot \tilde{\mathbf{h}}$. Hence, the nonsingularity of $H(\tilde{\mathbf{n}}) \cdot \tilde{\mathbf{h}}$ implies that of $(\dfrac{\partial^2 F}{\partial u_i\, u_j}(\mathbf{x}_0, u_0))$. □

Although Proposition 5.1 handles the case of a smooth manifold in $\mathcal{R}$, we saw in §3 that usually the image in $\mathcal{R}$ of a generic hypersurface M in $\mathbb{R}^n$ will have Legendrian singularities and the image itself is a Whitney stratified set $\tilde{M}$. Next, we deduce the condition ensuring that the envelope is smooth at a singular point $\mathbf{x}_0$.

Because $\tilde{M}$ has Legendrian singularities, it has a special property. To expain it we use a special property which holds for certain Whitney stratified sets.

Definition 5.2. An m-dimensional Whitney stratified set $M \subset \mathbb{R}^k$ has the *Unique Limiting Tangent Space Property* (ULT property) if for any $x \in M_{sing}$, a singular point of M, there is a unique m-plane $\Pi \subset \mathbb{R}^k$ such that for any sequence $\{x_i\}$ of smooth points in M_{reg} such that $\lim x_i = x$, we have $\lim T_{x_i} M = \Pi$

Lemma 5.3. *For a generic hypersurfaces $M \subset \mathbb{R}^n$, if $z \in \tilde{M}$, then $\tilde{M}$ can be locally represented in a neighborhood of z as a finite transverse union of $(n-1)$-dimensional Whitney stratified sets Y_i each having the ULT property.*

Transverse union means that if W_{ij} is the stratum of Y_i containing z than the W_{ij} intersect transversally.

Proof. The Lemma follows because $\tilde{M}$ consists of generic Legendrian singularities, which are either stable (or topologically stable) Legendrian singularities. These are either discriminants of stable unfoldings of multigerms of hypersurface singularities or transverse sections of such. Such discriminants are transverse unions of discriminants of individual hypersurface singularities, each of which have the ULT property by a result of Saito [Sa]. This continues to hold for transverse sections. □

We shall refer to these as the *local components* of $\tilde{M}$ in a neighborhood of z. There is then a corollary of the preceding.

Corollary 5.4. *Suppose that $\tilde{M}$ is an $(n-1)$–dimensional Whitney stratified set in $\mathcal{R}$ such that: at every smooth point z of $\tilde{M}$, the hypotheses of Proposition 5.1 holds; and $\tilde{M}$ is at all singular points locally the finite union of Whitney stratified sets Y_i each having the ULT property. Then,*

i) *The envelope of M of $\tilde{M}$ has a unique point $x \in M$ for each $z \in \tilde{M}_{reg}$, and M is smooth at all points corresponding to points in $\tilde{M}_{reg}$.*
ii) *At each singular point z of $\tilde{M}$, there is a point in M corresponding to each local component of $\tilde{M}$ in a neighborhood of z.*

Proof. First, if $z \in \tilde{M}_{reg}$ and satisfies the conditions of Proposition 5.1, then there is a unique envelope point corresponding to z and the envelope is smooth at that point.

Second, via the isomorphism $\tilde{\nu}$ and the commutative diagram (3.1), the envelope construction corresponds to the inverse $\delta^\vee$ of δ (or rather a local version since we have an orientation). Under the isomorphism $\tilde{\nu}$, for each point $z \in \tilde{M}_{sing}$ there corresponds a unique point in the envelope for each local component of $\tilde{M}$ containing z. It is obtained as $\delta^\vee$ applied to the unique limiting tangent space of z associated to the local component in $\tilde{M}_{reg}$. □

6. **Induced Geodesic Flow between Hypersurfaces**

We can bring together the results of the previous sections to define the Lorentzian geodesic flow between two smooth generic hypersurfaces with a correspondence. We denote our hypersurfaces by M_0 and M_1 and let $\chi : M_0 \to M_1$ be a diffeomorphism giving the correspondence. Note that we allow the hypersurfaces to have boundaries.

We suppose that both are oriented with unit normal vector fields $\mathbf{n}_0$ and $\mathbf{n}_1$. We also need to know that they have a "local relative orientation".

Definition 6.1. We say that the oriented manifolds M_0 and M_1, with unit normal vector fields $\mathbf{n}_0$ and $\mathbf{n}_1$, and with correspondence $\chi : M_0 \to M_1$ are *relatively oriented* if for each $x_0 \in M_0$, $\mathbf{n}_0(x_0) \neq -\mathbf{n}_1(\chi(x_0))$.

Theorem 6.2 (Existence, Smooth Dependence and Stability of Lorentzian Geodesic Flows).
Suppose smooth generic hypersurfaces M_0 and M_1 are oriented by smooth unit normal vector fields $\mathbf{n}_i, i = 0, 1$ and are relatively oriented for the diffeomorphism χ.

(1) (Existence and Smoothness:) *There is a unique Lorentzian geodesic flow ψ_t between $\tilde{M}_0 = M_{0\,\mathcal{L}}$ and $\tilde{M}_1 = M_{1\,\mathcal{L}}$ which is smooth.*
(2) (Stability:) *There is a neighborhood $\mathcal{U}$ of χ in $Diff(M_0, M_1)$ (for the C^∞–topology) such that if $\chi' \in \mathcal{U}$, then M_0 and M_1 are relatively oriented for χ' and the map $\Psi : \mathcal{U} \to C^\infty(M_0 \times [0, 1], \mathcal{R})$ mapping χ' to the associated Lorentzian flow $\tilde{\psi}'_t$ is continuous.*
(3) (Smooth Dependence:) *Let $\chi_s : M_{0\,s} \to M_{1\,s}$ be a smooth family of diffeomorphisms between smooth families of hypersurfaces for $s \in S$, a smooth manifold (i.e. $M_{i\,s}$ is the image of $M_i \times S$ under a smooth family of embeddings) so that $M_{0\,s}$ and $M_{1\,s}$ are relatively oriented for χ_s for each s. Then, the family of Lorentzian Geodesic flows $\tilde{\psi}_{s,t}$ between $\tilde{M}_{0\,s}$ and $\tilde{M}_{1\,s}$ is a smooth function of s (and x and t).*

Proof. For $x \in M_0$, suppose the tangent space $T_x M_0$ is defined by $\mathbf{n}_0(x) \cdot \mathbf{x} = c_0(x)$, and similarly $T_y M_1$ is defined by $\mathbf{n}_0(y) \cdot \mathbf{x} = c_1(y)$. It follows from relative orientation that for each x_0, there is a unique shortest geodesic $(\mathbf{n}_t(x_0), c_t(x_0))$ in Λ^{n+1} from $(\mathbf{n}_0(x_0), c_0(x_0))$ to $(\mathbf{n}_1(\chi(x_0)), c_1(\chi(x_0)))$.

First, to establish the smoothness of the geodesic flow, we note that by (6) of (2.1) if $\mathbf{n}_0(x_0) \neq -\mathbf{n}_1(\chi(x_0))$, then there is a neighborhood $x_0 \in W \subset M_0$ where the shortest geodesic between $(\mathbf{n}_0(x), c_0(x)\boldsymbol{\epsilon})$ and $(\mathbf{n}_1(\chi(x)), c_1(\chi(x))\boldsymbol{\epsilon})$ depends smoothly on the end points. Here for $x \in M_0$, we suppose the tangent space $T_x M_0$ is defined by $\mathbf{n}_0(x) \cdot \mathbf{x} = c_0(x)$, and similarly $T_y M_1$ is defined by $\mathbf{n}_0(y) \cdot \mathbf{x} = c_1(y)$.

Hence, the Lorentzian flow is locally smooth and by the relative orientation, it is well–defined everywhere. Hence it is a globally smooth well- defined flow between $(\mathbf{n}_0(x), c_0(x)\boldsymbol{\epsilon})$ and $(\mathbf{n}_1(\chi(x)), c_1(\chi(x))\boldsymbol{\epsilon})$ for each $x \in M_0$.

For smooth dependence, we use an analogous argument. Given the unique geodesic joining $(\mathbf{n}_{0\,s_0}(x_0), c_{0\,s_0}(x_0)\boldsymbol{\epsilon})$ and $(\mathbf{n}_{1\,s_0}(\chi_{s_0}(x_0)), c_{1\,s_0}(\chi_{s_0}(x_0))\boldsymbol{\epsilon})$, then there exists a neighborhood W of (x_0, s_0) so that for $(x, s) \in W$ there is a unique minimal geodesic between $(\mathbf{n}_{1\,s}(\chi_s(x)), c_{1\,s}(\chi_s(x))\boldsymbol{\epsilon})$ and $(\mathbf{n}_{1\,s}(\chi_s(x)), c_{1\,s}(\chi_s(x))\boldsymbol{\epsilon})$, and the geodesics depend smoothly on (x, s).

Thus, the global Lorentzian geodesic flow is uniquely defined and locally depends smoothly on (x, s); hence so does the global flow.

Finally to establish the stability, given χ for which M_0 and M_1 are relatively oriented, the set $U = \{(x, y) \in M_0 \times M_1 : \mid \mathbf{n}_0(x) \cdot \mathbf{n}_1 \mid > 0\}$ is an open set. Hence, as M_0 and M_1 are compact,

$$\mathcal{U} \;=\; \{\chi' \in Diff(M_0, M_1) : \{(x, \chi'(x)) : x \in M_0\} \subset U\}$$

is an open set for the C^∞–topolopy.

Second, given $\chi' \in \mathcal{U}$, consider the mapping $\chi'_{\mathcal{L}} : M_0 \to \mathcal{R} \times \mathcal{R}$ defined by $x \mapsto ((\mathbf{n}_0(x), c_0(x)), (\mathbf{n}_1(x), c_1(x))$, where $(\mathbf{n}_0(x), c_0(x)$ defines the tangent space $T_x M_0$ and $(\mathbf{n}_1(x), c_1(x)$ defines the tangent space $T_{\chi'(x)} M_1$. $\chi'_{\mathcal{L}}$ is defined using the first derivatives of the embeddings $M_i \subset \mathbb{R}^n$ and χ' composed with algebraic operations. Each such operation is continous in the C^∞–topology and so defines a continuous map $\mathcal{L}' : \mathcal{U} \to C^\infty(M_0, \mathcal{R} \times \mathcal{R})$. Lastly, the Lorentzian flow ψ_t is defined by (4.4), and is the composition of $\mathcal{L}'$ with algebraic operations involving the smooth functions $\lambda(x, \theta)$, and is again continuous in the C^∞–topology. Hence, the combined composition mapping $\chi\prime \to \psi_t$ is continuous in the C^∞–topology. □

Remark . We note there are two consequences of 2) of Theorem 6.2. First, M_0 and M_1 may remain fixed, but the correspondence χ varies in a family. Then the corresponding Lorentzian geodesic flows vary in a family. Second, M_0 and M_1 may vary in a family with a corresponding varying correspondence, then the Lorentzian geodesic flow will also vary smoothly in a family.

It remains to determine when the corresponding Lorentzian geodesic flows in R^n will have analogous properties.

We consider the vector fields on M_0, $\mathbf{n}_0(x)$ and $\mathbf{n}_1(\chi(x))$. For any vector field $\mathbf{n}(x)$ on M_0 with values in $\mathbb{R}^n$, we let $N(x) = (\mathbf{n}(x) \,|\, d\mathbf{n}(x))$ be the $n \times n$ matrix with columns $\mathbf{n}(x)$ viewed as a column vector and $d\mathbf{n}(x)$ the $n \times (n-1)$ Jacobian matrix.. If we have a local parametrization $\mathbf{x}(u)$ of M_0, then we may represent the vector field $\mathbf{n}$ as a function of u, $\mathbf{n}_(u)$. Then, $N(\mathbf{x}(u))$ is the $n \times n$ matrix with columns $\mathbf{n}(u), \mathbf{n}_{u_1}(u), \dots, \mathbf{n}_{u_{n-1}(u)}$. We denote the matrix for $\mathbf{n}_0$ by $N_0(x)$, and that for $\mathbf{n}_1(\chi(x))$ by $N_1(x)$ (or $N_0(u)$ and $\mathbf{n}_1(\chi(u))$ if we have parametrized M_0.

Consider the Lorentzian geodesic flow $\tilde{\psi}_t(x) = (\mathbf{n}_t(x), c_t(x))$ between $\mathcal{L}(x) = (\mathbf{n}_0(x), c_0(x))$ and $\mathcal{L}(\chi(x)) = (\mathbf{n}_1(\chi(x)), c_1(\chi(x)))$ for all $x \in M_0$. We let $\tilde{M}_t = \tilde{\psi}_t(M_0)$, and we let M_t denote the envelope of $\tilde{M}_t$.

We introduce one more function.

$$\sigma(x, \theta) \;=\; \frac{\cos((1-x)\theta)\sin(x\theta) \;-\; x \sin\theta}{\sin(x\theta)\, \sin\theta} \;=\; \frac{\cos((1-x)\theta)}{\sin\theta} \;-\; \frac{x}{\sin(x\theta)}$$

if $0 < |\theta| < \pi$, and

$$\sigma(x, 0) \;=\; 0$$

Then there are the following properties for the envelopes M_t of the flow for all time $0 \leq t \leq 1$.

Theorem 6.3. *Suppose smooth generic hypersurfaces M_0 and M_1 are oriented by smooth unit normal vector fields $\mathbf{n}_i, i = 0, 1$ and are relatively oriented. Let $\tilde{\psi}_t$ be the Lorentzian geodesic flow between $\tilde{M}_0$ and $\tilde{M}_1$ is smooth. If M_t is the family of envelopes obtained from the flow $\tilde{M}_t = \tilde{\psi}_t(\tilde{M}_0)$, then suppose that for each time t, $\tilde{M}_t$ has only generic Legendrian singularities as in §3 (as e.g. in Fig. 4). Then,*

(1) *M_t will have a unique point corresponding to $z = \tilde{\psi}_t(x) \in \tilde{M}_t$ provided*

$$N'_t(x) \overset{def}{=} \lambda(t,\theta)\, N_1(x) \,+\, \lambda(1-t,\theta)\, N_0(x) \,+\, \sigma(t,\theta)\, \frac{\partial \theta}{\partial \mathbf{u}}\, \mathbf{n}_0 \tag{6.1}$$

is nonsingular. Here $\frac{\partial \theta}{\partial \mathbf{u}}\, \mathbf{n}_0$ is the matrix whose first column equals the vector 0 and whose $j{+}1$–th column is the vector $\frac{\partial \theta}{\partial u_j}\, \mathbf{n}_0$, for $j = 1, \dots, n-1$.

(2) *The envelope M_t will be smooth at points corresponding to a smooth point $z \in \tilde{M}_t$ satisfying (6.1) provided $H(\tilde{\mathbf{n}}_t(x))\cdot\tilde{\mathbf{h}}_t(x)$ is nonsingular. Here $\tilde{\mathbf{h}}_t(x)$ is defined from $\tilde{\mathbf{n}}_t(x)$ as in §5.*

(3) *At points corresponding to singular points $z \in \tilde{M}_t$, there is a unique point on M_t for each local component of $\tilde{M}$ in a neighborhood of z. This point is the unique limit of the envelope points corresponding to smooth points of the component of $\tilde{M}_t$ approaching z.*

Proof of Theorem 6.3 . For 2), given that 1) holds, we may apply ii) of Proposition 5.1. For 3) we may apply Corollary 5.4. To prove 1), we will apply i) of Proposition 5.1. We must give a sufficient condition that $N(x)$ is nonsingular. We choose local coordinates u for a neighborhood of $\mathbf{x}_0$. For a geodesic $(\mathbf{n}_t(u), c_t(u)\boldsymbol{\epsilon})$ between $(\mathbf{n}_0(u), c_0(u)\boldsymbol{\epsilon})$ and $(\mathbf{n}_1(u), c_1(u)\boldsymbol{\epsilon})$ given by (4.4), we must compute $\mathbf{n}_{t\, u_i}(u)$. We note that not only $\mathbf{n}_i, i = 1, 2$ but also θ depends on u. We obtain

$$\mathbf{n}_{t\, u_i} \;=\; \lambda(t,\theta)\, \mathbf{n}_{1\, u_i} \,+\, \lambda(1-t,\theta)\, \mathbf{n}_{0\, u_i} \,+\, \frac{\partial \lambda(t,\theta)}{\partial u_i}\, \mathbf{n}_1 \,+\, \frac{\partial \lambda(1-t,\theta)}{\partial u_i}\, \mathbf{n}_0 \tag{6.2}$$

Then, $\frac{\partial \lambda(t,\theta)}{\partial u_i} = \frac{\partial \theta}{\partial u_i}\, \frac{\partial \lambda(t,\theta)}{\partial \theta}$. First suppose $\theta \neq 0$, then we compute

$$\frac{\partial \lambda(x,\theta)}{\partial \theta} \;=\; \frac{x\, \sin(\theta)\cos(x\theta) \,-\, \sin(x\theta)\cos\theta}{\sin^2\theta} \tag{6.3}$$

Applying (6.3) with $x = t$ and $1-t$, we obtain for the last two terms on the RHS of (6.2)

$$\begin{aligned} \frac{\partial \lambda(t,\theta)}{\partial u_i}\, \mathbf{n}_1 \,+\, \frac{\partial \lambda(1-t,\theta)}{\partial u_i}\, \mathbf{n}_0 \;=\; \frac{\partial \theta}{\partial u_i}\Big(&\frac{t\, \cos(t\,\theta)}{\sin\theta}\mathbf{n}_1 \,+\, \frac{(1-t)\, \cos((1-t)\,\theta)}{\sin\theta}\mathbf{n}_0 \\ &- \cot\theta\, (\lambda(t,\theta)\, \mathbf{n}_1 \,+\, \lambda(1-t,\theta)\, \mathbf{n}_0)\Big) \end{aligned} \tag{6.4}$$

We see that the last expression in (6.4) is a multiple of $\mathbf{n}_t$. We can subtract a multiple of $\mathbf{n}_t$ from $\mathbf{n}_{t\, u_i}$ without altering the rank of the matrix N_t. Then, after subtracting $\frac{\partial \theta}{\partial u_i}\, \cot\theta\, \mathbf{n}_t$ from the RHS of (6.4), we obtain

$$\frac{\partial \theta}{\partial u_i}\Big(\frac{t\, \cos(t\,\theta)}{\sin\theta}\mathbf{n}_1 \,+\, \frac{(1-t)\, \cos((1-t)\,\theta)}{\sin\theta}\mathbf{n}_0\Big) \tag{6.5}$$

Then, in addition, we can subtract $\frac{\partial \theta}{\partial u_i}\, t\cot(t\,\theta)\, \mathbf{n}_t$ from the RHS of (6.5) so the term involving $\mathbf{n}_1$ is removed. We are left with

$$\frac{\partial \theta}{\partial u_i}\,\Big(\frac{(1-t)\, \cos((1-t)\,\theta)}{\sin\theta} \,-\, t\, \cot(t\theta)\, \frac{\sin((1-t)\theta)}{\sin\theta}\Big)\, \mathbf{n}_0 \tag{6.6}$$

Adding the two terms in the parentheses in (6.6), rearranging, and using the formula for $\sin(A-B)$, we obtain $\sigma(t,\theta)$, so that (6.6) becomes $\dfrac{\partial\theta}{\partial u_i}\,\sigma(t,\theta)\,\mathbf{n}_0$. Thus, applying the preceding to each $\mathbf{n}_{t\,u_i}$ we may replace each of them with

$$\lambda(t,\theta)\,\mathbf{n}_{1\,u_i} \;+\; \lambda(1-t,\theta)\,\mathbf{n}_{0\,u_i} \;+\; \frac{\partial\theta}{\partial u_i}\,\sigma(t,\theta)\,\mathbf{n}_0$$

without changing the rank. We conclude that N_t has the same rank as the matrix N'_t given in (6.1). □

Remark . If $\mathbf{n}_1(\chi(x_0)) \neq \mathbf{n}_0(x_0)$, then there is a neighborhood $x_0 \in W \subset M_0$ such that $\mathbf{n}_1(\chi(x)) \neq \mathbf{n}_0(x)$ for $x \in W$. Then, there is a smooth unit tangent vector field $\mathbf{w}$ defined on W such that $\mathbf{n}_1(\chi(x))$ lies in the vector space spanned by $\mathbf{n}_0(x)$ and $\mathbf{w}(x)$, and $\mathbf{n}_1(\chi(x))\cdot\mathbf{w}(x) \geq 0$ for all $x \in W$. Then, smoothness follows explicitly using the geodesics given in (4.4).

7. **Flows in Special Cases**

We determine the form of the Lorentzian geodesic flow in several special cases.

Hypersurfaces Obtained by a Translation. Suppose that we obtain M_1 from M_0 by translation by a vector $\mathbf{p}$ and the correspondence associates to $\mathbf{x} \in M_0$, $\mathbf{x}+\mathbf{p} \in M_1$. Let $\mathbf{n}_0$ be a smooth unit normal vector field on M_0. The derivative of the translation map is the identity; hence, under translation $\mathbf{n}_0$ is mapped to itself translated to $\mathbf{x}' = \mathbf{x}+\mathbf{p}$. Thus, under the correspondence, $\mathbf{n}_1 = \mathbf{n}_0$. Also, If $\mathbf{n}_0\cdot\mathbf{x} = c_0$ is the equation of the tangent plane for M_0 at a point $\mathbf{x}$, then the tangent plane of M_1 at the point $\mathbf{x}'$ is

$$\mathbf{n}_1\cdot\mathbf{x}' \;=\; \mathbf{n}_0\cdot(\mathbf{x}+\mathbf{p}) \;=\; c_0 + \mathbf{n}_0\cdot\mathbf{p}$$

Hence, $c_1 = c_0 + \mathbf{n}_0\cdot\mathbf{p}$.

As $\mathbf{n}_0 = \mathbf{n}_1$, $\theta = 0$. Thus the geodesic flow on $\mathcal{R}$ is given by

$$t(\mathbf{n}_0, c_1\boldsymbol{\epsilon}) + (1-t)(\mathbf{n}_0, c_0\boldsymbol{\epsilon}) \;=\; (\mathbf{n}_0, c_0\boldsymbol{\epsilon}) + (0, (t\mathbf{n}_0\cdot\mathbf{p})\boldsymbol{\epsilon}) \;=\; (\mathbf{n}_0, (\mathbf{n}_0\cdot(\mathbf{x}+t\mathbf{p}))\boldsymbol{\epsilon})$$

Thus, at time t the tangent space is translated by $t\mathbf{p}$. Thus the envelope of these translated hyperplanes is the translation of M_0 by $t\mathbf{p}$. Hence, we conclude

Corollary 7.1. *If M_1 is the translation of M_0 by $\mathbf{p}$, then the Lorentzian geodesic flow is translation by $t\mathbf{p}$.*

Second we consider the case of a homothety.

Hypersurfaces Obtained by a Homothety. Suppose that we obtain M_1 from M_0 by multiplication by a constant b and the correspondence associates to $\mathbf{x} \in M_0$, $\mathbf{x}' = c\mathbf{x} \in M_1$. The derivative of the multiplication map by b is multiplication by b; hence, under the multiplication map $T_{\mathbf{x}}M_0$ is mapped to $T_{\mathbf{x}'}M_1$. If $\mathbf{n}_0$ is a smooth unit normal vector field on M_0, then $\mathbf{n}_0$ remains normal to $T_{\mathbf{x}'}M_1$. Hence, $\mathbf{n}_1 = \mathbf{n}_0$ translated to $\mathbf{x}'$. Also, if $\mathbf{n}_0\cdot\mathbf{x} = c_0$ is the equation of the tangent plane for M_0 at a point $\mathbf{x}$, then the tangent plane of M_1 at the point $\mathbf{x}'$ is

$$\mathbf{n}_1\cdot\mathbf{x}' \;=\; \mathbf{n}_0\cdot(b\mathbf{x}) \;=\; bc_0$$

Hence, $c_1 = bc_0$.

Again $\mathbf{n}_0 = \mathbf{n}_1$ so $\theta = 0$. Thus the geodesic flow on $\mathcal{R}$ is given by

$$t(\mathbf{n}_0, c_1\boldsymbol{\epsilon}) + (1-t)(\mathbf{n}_0, c_0\boldsymbol{\epsilon}) \; = \; (\mathbf{n}_0, (tb + (1-t))c_0\boldsymbol{\epsilon})$$

Thus, at time t the tangent plane is transformed by multiplication by $(tb+(1-t))$. Thus the envelope of these hyperplanes is M_0 multiplied by $(tb + (1-t))$. Hence, we conclude

Corollary 7.2. *If M_1 is obtained from M_0 by multiplication by the constant b, then the Lorentzian geodesic flow is the family of hypersurfaces obtained by applying to M_0 the family of homotheties, multiplication by $(tb + (1-t))$.*

Third, we consider the case of a rotation.

Hypersurfaces Obtained by a Rotation. Suppose that we obtain M_1 from M_0 by a rotation A about the origin in a plane (which pointwise fixes an orthogonal subspace. Choosing coordinates, we may assume that the rotation A is in the (x_1, x_2)–plane and rotates by an angle ω. We also suppose the correspondence associates to $\mathbf{x} \in M_0$, $\mathbf{x}' = A(\mathbf{x}) \in M_1$. Consider a tangent space at $\mathbf{x} \in M_0$, defined by $\mathbf{n}_0 \cdot \mathbf{x} = c_0$. As $A(\mathbf{n}_0) \cdot A(\mathbf{x}) = \mathbf{n}_0 \cdot \mathbf{x} = c_0$, if we let $\mathbf{x}' = A(\mathbf{x})$, then the equation of the tangent plane for M_1 at $\mathbf{x}'$ is defined by $A(\mathbf{n}_0) \cdot \mathbf{x}' = c_0$. Hence, $\mathbf{n}_1 = A(\mathbf{n}_0)$ and $c_1 = c_0$.

To express the geodesic flow, we write $\mathbf{n}_0 = \mathbf{v} + \mathbf{p}$ where $\mathbf{v}$ is in the rotation plane and $\mathbf{p}$ is fixed by A. Hence, $\mathbf{n}_1 = A(\mathbf{v}) + \mathbf{p}$. Thus, the angle θ between $\mathbf{n}_0$ and $\mathbf{n}_1$ satisfies

$$\cos\theta \; = \; \mathbf{n}_1 \cdot \mathbf{n}_0 \; = \; A(\mathbf{v}) \cdot \mathbf{v} \; + \; \mathbf{p} \cdot \mathbf{p}$$

As $\|\mathbf{n}_0\| = 1$, we obtain $\mathbf{v} \cdot \mathbf{v} \; + \; \mathbf{p} \cdot \mathbf{p} = 1$. Also, $A(\mathbf{v}) \cdot \mathbf{v} = \|\mathbf{v}\|^2 \cos\omega$. Hence,

$$\cos\theta \; = \; 1 + \|\mathbf{v}\|^2(\cos\omega - 1) \tag{7.1}$$

We recall that by (4.3)

$$\lambda(t, \theta) + \lambda(1-t, \theta) \; = \; \mu(1 - 2t, \frac{\theta}{2})$$

Using the expressions for $\mathbf{n}_0$ and $\mathbf{n}_1$, we find the geodesic flow is given by

$$\begin{aligned} &= \; \lambda(t, \theta)\,(A(\mathbf{n}_0), c_0\boldsymbol{\epsilon}) + \lambda(1-t, \theta)\,(\mathbf{n}_0, c_0\boldsymbol{\epsilon}) \\ &= \; ((\lambda(t, \theta)A(\mathbf{v}) + \lambda(1-t, \theta)\,\mathbf{v}) \; + \; \mu(1 - 2t, \frac{\theta}{2})\,\mathbf{p}, \; \mu(1 - 2t, \frac{\theta}{2})c_0\boldsymbol{\epsilon}) \qquad (7.2) \end{aligned}$$

We note that $\mu(1 - 2t, \frac{\theta}{2})$ is a function of t on $[0, 1]$ which has value $= 1$ at the end points, and has a maximum $= \sec(\frac{1}{2}\theta)$ at $t = \frac{1}{2}$. Thus, the geodesic flow $(\mathbf{n}_t, c_t\boldsymbol{\epsilon})$ has the contribution in the rotation plane given by $\lambda(t, \theta)A(\mathbf{v}) + \lambda(1-t, \theta)\mathbf{v}$ which is not a true rotation from $\mathbf{v}$ to $A(\mathbf{v})$. Also, the other contribution to $\mathbf{n}_t$ is from $\mu(1 - 2t, \frac{\theta}{2})\mathbf{p}$ which increases and then returns to size $\mathbf{p}$ (see Fig. 5). In addition, the distance from the origin will vary by $\mu(1 - 2t, \frac{\theta}{2})c_0$. These form a type of "pseudo rotation". This yields the following corollary.

Corollary 7.3. *If M_1 is obtained from M_0 by rotation in a plane (with fixed orthogonal complement), then the Lorentzian geodesic flow is the family of hypersurfaces obtained by applying to M_0 the family of pseudo rotations given by (7.2).*

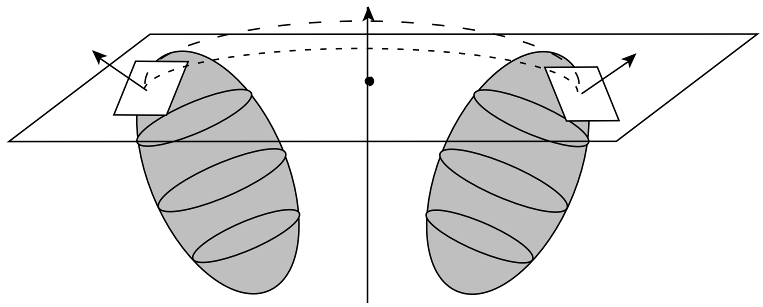

FIGURE 5. Lorentzian Geodesic Flow between a surface and a rotated copy is given by a "pseudo–rotation". The path of the rotation is indicated by the dotted curve, while that for the pseudo rotation is given by the broken curve, which lifts out of the plane of rotation before returning to it.

Invariance under Scalar Multiplication and Rigid Motions. We can use the calculations used in the preceding to establish the invariance of the Lorentzian geodesic flow under scalar multiplication and rigid motions.

Suppose Π is a hyperplane in $\mathbb{R}^n$ defined by $(\mathbf{n}, c)$. If ϕ is a transformation defined by: multiplication by b; respectively translation by $\mathbf{p}$; respectively orthogonal transformation A, then $\Pi' = \phi(\Pi)$ is defined by: $(\mathbf{n}, bc)$; respectively $(\mathbf{n}, c + \mathbf{n} \cdot \mathbf{p})$; respectively $(A(\mathbf{n}), c)$ Now suppose Π_t, defined by $\psi(t) = (\mathbf{n}_t, c_t)$, is a Lorentzian geodesic flow between Π_0 and Π_1.

Let ϕ be one of: multiplication by b; respectively translation by $\mathbf{p}$; respectively orthogonal transformation A. Let $\Pi'_t = \psi(\Pi_t)$. Then, by (4.4)

$$(\mathbf{n}_t, c_t) \;=\; (\lambda(t,\theta)\mathbf{n}_1 \,+\, \lambda(1-t,\theta)\,\mathbf{n}_0, \lambda(t,\theta)\,c_1 \,+\, \lambda(1-t,\theta)\,c_0) \tag{7.3}$$

First, in the case of multiplication by b, Π'_t is given by

$$(\mathbf{n}_t, bc_t) \;=\; (\lambda(t,\theta)\mathbf{n}_1 \,+\, \lambda(1-t,\theta)\,\mathbf{n}_0, \lambda(t,\theta)\,bc_1 \,+\, \lambda(1-t,\theta)\,bc_0) \tag{7.4}$$

which is the Lorentzian geodesic flow between $(\mathbf{n}_0, bc_0)$ and $(\mathbf{n}_1, bc_1)$.

An analogous argument works for the other two cases using the forms of Π' given above. As a general composition of scalar multiplication and rigid transformations is given as a composition of these three, the invariance follows.

8. Results for the Case of Surfaces in $\mathbb{R}^3$

Now we consider the special case of surfaces $M_i \subset \mathbb{R}^3$, i = 1, 2 for which there is a correspondence given by the diffeomorphism $\chi : M_0 \to M_1$. We suppose each M_i is a generic smooth surface with $\mathbf{n}_0 = (a_1, a_2, a_3)$ and $\mathbf{n}_1 = (a'_1, a'_2, a'_3)$ smooth unit normal vector fields on M_0, respectively M_1. We assume that $X(u_1, u_2)$ is a local parametrization of M_0. Also, let $\mathbf{n}_i(u) \cdot \mathbf{x} = c_i(u)$ define the tangent planes for M_0 at $X(u_1, u_2)$, .respectively M_1 at $\chi(X(u_1, u_2))$

We let

$$\mathbf{n}_t \;=\; (a_{1\,t}, a_{2\,t}, a_{3\,t}) \;=\; \lambda(t,\theta)\,(a'_1, a'_2, a'_3) \,+\, \lambda(1-t,\theta)\,(a_1, a_2, a_3)$$

and $c_t(u) = \lambda(t,\theta)\, c_1 \, + \, \lambda(1-t,\theta)\, c_0$. Then,

$$N_t \;=\; \begin{pmatrix} a_{1\,t} & a_{1\,t\,u_1} & a_{1\,t\,u_2} \\ a_{2\,t} & a_{2\,t\,u_1} & a_{2\,t\,u_2} \\ a_{3\,t} & a_{3\,t\,u_1} & a_{3\,t\,u_2} \end{pmatrix} \tag{8.1}$$

Remark . Note here and what follows we use the following notation. For quantities defined for a flow, we denote dependence on t by a subscript. We also want to denote partial derivatives with respect to the parameters u_i by a subscript. To distinguish them, the subscripts appearing after a comma will denote the partial derivatives. Hence, for example, in (8.1), $a_{i\,t,u_j} = \dfrac{\partial a_{i\,t}}{\partial u_j}$

Existence of Envelope Points. The sufficient condition that there is a unique point $X_{t_0}(u)$ in the Lorentzian geodesic flow in $\mathbb{R}^3$ at time $t = t_0$ is that (8.1) evaluated at $t = t_0$ and $u = (u_1, u_2)$ is nonsingular. Then, the unique point is the solution of the linear system.

$$N_{t_0}^T \cdot \mathbf{x} \;=\; \mathbf{c} \tag{8.2}$$

with $\mathbf{x}$ and $\mathbf{c}$ column matrices with entries x_1, x_2, x_3, respectively $c_{t_0}, c_{t_0,u_1}, c_{t_0,u_2}$.

Furthermore, the nonsingularity of (8.1) is equivalent to that

$$N'_{t_0} \;=\; \lambda(t_0,\theta)\, N_1 \, + \, \lambda(1-t_0,\theta)\, N_0 + \sigma(t_0,\theta)\, \frac{\partial \theta}{\partial \mathbf{u}}\, \mathbf{n}_0 \tag{8.3}$$

where

$$\frac{\partial \theta}{\partial \mathbf{u}}\, \mathbf{n}_0 \;=\; \begin{pmatrix} 0 & \theta_{u_1} a_1 & \theta_{u_2} a_1 \\ 0 & \theta_{u_1} a_2 & \theta_{u_2} a_2 \\ 0 & \theta_{u_1} a_3 & \theta_{u_2} a_3 \end{pmatrix} \tag{8.4}$$

Smoothness of the Envelope. For the smoothness of M_{t_0} at the point $X_{t_0}(u_1, u_2)$, we let

$$\tilde{\mathbf{n}}_{t_0} = (a_{1\,t_0}, a_{2\,t_0}, a_{3\,t_0}, -c_{t_0})$$

evaluated at $u = (u_1, u_2)$. Also, we let $\tilde{h} = \mathbf{n}_{t_0} \times \mathbf{n}_{t_0\,u_1} \times \mathbf{n}_{t_0\,u_1}$, which is the analogue of the cross product but for vectors in $\mathbb{R}^4$. It is the vector whose j–th entry is $(-1)^{j+1}$ times by taking the 3×3 determinant of the submatrix obtained by deleting the j–th column of

$$\begin{pmatrix} a_{1\,t_0} & a_{2\,t_0} & a_{3\,t_0} & -c_{t_0} \\ a_{1\,t_0,u_1} & a_{2\,t_0,u_1} & a_{3\,t_0,u_1} & -c_{t_0,u_1} \\ a_{1\,t_0,u_2} & a_{2\,t_0,u_2} & a_{3\,t_0,u_2} & -c_{t_0,u_2} \end{pmatrix} \tag{8.5}$$

Then, we form the 2×2–matrix $H(n_t(u)) \cdot \tilde{\mathbf{n}}_t(u)$ with ij–th entry $n_{t,u_i u_j}(u) \cdot \tilde{\mathbf{h}}(u)$ for $u = (u_1, u_2)$. Then, from Theorem 6.3, we conclude that for a point uniquely defined by (8.2) the envelope is smooth at $X_{t_0}(u)$ if $H(n_{t_0}(u)) \cdot \tilde{\mathbf{n}}_{t_0}(u)$ is nonsingular.

Envelope Points corresponding to Legendrian Singular Points. Third, the generic Legendrian singularities for surfaces are those given in Fig. 4). For these:

1. (1) At points on cuspidal edges or swallowtail points $z \in \tilde{M}_t$, there is a unique point on M_t which is the unique limit of the envelope points corresponding to smooth points of $\tilde{M}_t$ approaching z.

(2) At points $z \in \tilde{M}_t$ which are tranverse intersections of two or three smooth $(n-1)$-dimensional submanifolds, or the transverse intersection of a smooth manifold ans a cuspidal edge, there is a unique point in M_t for each smooth $(n-1)$-dimensional submanfold passing through z (and one for the cuspidal edge).

References

[A1] Arnol'd, V. I. , *Singularities of Systems of Rays*, Russian Math. Surveys **38 no. 2** (1983), 87–176

[AGV] Arnold, V. I., Gusein-Zade, S. M., Varchenko, A. N., *Singularities of Differentiable Maps*, Volumes 1, 2 (Birkhauser, 1985).

[BMTY] Beg M. F., Miller M. I., Trouvé A., and Younes L., *Computing large deformation metric mappings via geodesic flows of diffeomorphisms*, Int. Jour. Comp. Vision, **61** (2005), 139–157.

[B1] Bruce, J. W., *The Duals of Generic Hypersurfaces*, Math. Scand. **49** (1981) 36–60

[BG1] Bruce, J. W., Giblin, P. J., *Curves and Singularities*, 2nd edn., Cambridge University Press, (1992).

[D1] Damon, J. *Smoothness and Geometry of Boundaries Associated to Skeletal Structures I: Sufficient Conditions for Smoothness*, Annales Inst. Fourier **53 no.6** (2003) 1941-1985

[D2] ——— *Swept Regions and Surfaces: Modeling and Volumetric Properties*, to appear Theoretical Comp. Science, Conf Comp. Alg. Geom, Nice '06, in honor of Andre Galligo

[MM] Mumford, D. and Michor, P. *Riemannian geometries on spaces of plane curves*, Jour. European Math. Soc., **Vol. 8, No. 1**, 2006.

[MM2] ——— *An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach*, to appear Applied and Computational Harmonic Analysis (ACHA). arXiv:math.DG/0605009.

[Ma1] Mather, J.N., *Generic Projections*, Annals of Math. **vol 98** (1973) 226–245

[Ma2] ———, *Solutions of Generic linear Equations*, in Dynamical Systems, M. Peixoto, Editor, (1973) Academic Press, New York, 185–193

[Ma3] ———, *Notes on Right Equivalence*, unpublished preprint

[OH] O'Hara, J., *Energy of Knots and Conformal Geometry*, Series on Knots and Everything **Vol. 33**, World Scientific Publ., New Jersey-London-Singapore-Hong Kong (2003)

[ON] O'Neill, B., *Semi-Riemannian Geometry: with Applications to Relativity*, Series in Pure and Applied Mathematics, Academic Press, (1983)

[S] Saito, K.,

[Tr] Trouvé A., *Diffeomorphism groups and pattern matching in image analysis*, Int. Jour. Comp. Vision, **28** (1998), 213–221.

[YTG] Glaunés, J., Trouvé, A., and Younes, L. *Diffeomorphic matching of distributions: A new approach for unlabelled point-sets and sub-manifolds matching* , in Proc. CVPR04, 2004.

[YTG2] ——— *Modeling Planar Shape Variation via Hamiltonian Flows of Curves*, (2005) preprint

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