Chapter 1
Jim Damon's contributions to singularity theory and its applications

Bill Bruce, Peter Giblin, David Mond, Stephen Pizer and Les Wilson

Abstract This Chapter presents a summary, with detailed references, of the work of James Damon (1945-2022) whose untimely death deprived the mathematical community of a major figure and, equally, of an exceptionally kind and supportive friend and colleague. In §1.1 his contributions to singularity theory as a pure mathematical discipline are covered, including major extensions of René Thom and John Mather's theories of smooth and topological equivalence of mappings, stability, sufficiency and versal unfoldings; Milnor fibrations; free and almost free divisors; coherence; bifurcation theory; and partial differential equations. In §1.2 Jim's work on medial and skeletal structures is covered; and in §1.3 his work on the interpretation in images of features, shade, shadow curves and apparent contours, and the manner in which these interact during viewer movement. Finally in §1.4 Jim's many contributions to medical image computing are described. He co-advised students and co-wrote many articles with computer scientists on such research topics as the statistics of shape of anatomic objects; segmentation by means of "height ridges" and "connector curves"; skeletal models as an aid to shape statistics; and generalised cylinders as representative of tube-like objects.

A note on references: Publications by Jim Damon, some joint with other authors, are collected separately in §1.6 and referred to by "Paper n", where

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Fig. 1.1 Jim Damon with his beloved wife Joanne, Chapel Hill June 2013

n is a positive integer. Other references, to works where Jim is not one of the authors, are collected at the end of the chapter, and are referred to in the usual way by citations of the form [XY]. Finally there is a list of mathematics PhDs supervised by Jim in §1.6.1 and a list of computer science dissertations co-advised by Jim Damon in §1.6.2. These are referred to by the item number within the appropriate subsection.

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1.1 Singularity theory: Bill Bruce, David Mond and Les Wilson

James Norman Damon (known to everyone as Jim) has been a leading figure in Singularity Theory since finishing his Doctoral Dissertation at Harvard University in 1972, under the supervision of John Mather. Jim passed away on October 5, 2022.

Initially largely focused on singularity theory itself, though with some early contributions to PDEs and Gaussian blurring, his later papers included more applications, especially in the area of computer vision. What is striking, reading through his bibliography, is the breadth of his interests, and the fact that so many of his papers marked a substantial and often deeply original contribution to the subject. From the first (results on the Gysin homomorphism for flag bundles) through to his last few papers (Milnor fibres of singular matrices, and medial linking structures in computer vision), it is clear that he had a deep understanding of large areas of mathematics, and huge originality, and continued to produce work of the highest quality right to the end. While many have made major contributions, Jim is one of the most significant influencers in taking our subject from those heady days of the early 70's to its present state. Naturally we cannot do full justice to a lifetime's achievements, but we can select some characteristic highlights.

For the benefit of those new to Singularity Theory, we begin with a brief discussion of basic concepts and of earlier work in the theory leading up to Jim's work. We try to only cover enough to aid in the understanding of those works that we cover. Much more detailed and comprehensive accounts of Singularity Theory can be found in for example [GG], [MN], [MN2] or [Ru]. Those experienced in Singularity Theory may well skip to the subsections below on Jim's papers.

A smooth (C^∞) mapping between smooth manifolds is said to be "stable" if "nearby" mappings are "equivalent" to it. To make this precise, one needs to define a topology on the set of all mappings (the "Whitney topology"), and an equivalence relation: two maps $f, g : N \to P$ are *right-left equivalent* (or $\mathcal{A}$-equivalent) if there are smooth diffeomorphisms $h : N \to N$ and $k : P \to P$ such that $g \circ h = k \circ f$. Then f is *stable* (or *right-left stable* or $\mathcal{A}$ stable) if there is a neighborhood of f such that all g in this neighborhood are right-left equivalent to f. For example, when $N = P = \mathbb{R}$, $f(x) = x^2$ is stable, but $g(x) = x^3$ is not since every neighborhood of g contains functions with no critical points as well as functions with two critical points, and those functions are not equivalent to g. A smooth mapping f is called *topologically stable* if we require only homeomorphisms instead of diffeomorphisms. The map $g(x) = x^3$ is not even topologically stable because g is one-to-one, but every neighborhood of g contains functions which are not one-to-one, and hence are not topologically equivalent to g. Some other notions of smooth equivalence are: f and g are *right* or $\mathcal{R}$-equivalent if there is a diffeomorphism h such that $g \circ h = f$; f and g are *left* or $\mathcal{L}$-equivalent if there is a diffeomorphism k such that $g = k \circ f$.

We can also define $\mathcal{A}$, $\mathcal{R}$ and $\mathcal{L}$ equivalence of germs of mappings $f, g : N, x_0 \to P, y_0$ by requiring h, k to be germs of diffeomorphisms at x_0 and y_0 respectively. We also allow multi-germs, where the single base-point x_0 is replaced by a finite set S of base-points. This is essential in order to consider questions of stability: for example, a germ of immersion $f : (N, x_0) \to (P, y_0)$ is stable, but a (global) immersion may fail to be stable because it has a self-tangency bi-germ $f : (N, \{x_0, x_1\}) \to (P, y_0)$, that is the germs of f at x_0 and x_1 are not transverse to one another.

A further equivalence relation, *contact equivalence* or $\mathcal{K}$-*equivalence* plays an important role in the theory, despite having less evident geometric significance: f and g are contact-equivalent if there are germs of diffeomorphisms $h : N, x_0 \to N, x_0$ and $K : N \times P, (x_0, y_0) \to N \times P, (x_0, y_0)$ of the form $K(x, y) = (h(x), k_1(x, y))$ with $k_1(x, y_0) = y_0$ such that K maps the graph of f onto the graph of g. Contact equivalence of f and g can be interpreted as saying that their graphs have the same contact with $N \times y_0$. Let us assume N, x_0 is $\mathbb{R}^n, 0$ and P, y_0 is $\mathbb{R}^p, 0$. Let $\mathcal{E}_n$ denote the ring of smooth germs $f : \mathbb{R}^n, 0 \to \mathbb{R}$ with maximal ideal $\mathfrak{m}_n$ (those smooth germs with $f(0) = 0$). The *ideal* of a map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is the ideal $I(f)$ in $\mathcal{E}_n$ generated by the components of $f : f^1, \ldots, f^p$. The local algebra of f is $\mathcal{Q}(f) = \mathcal{E}_n/I(f)$. It is easy to show that two map-germs are contact equivalent if, and only if, their algebras are isomorphic (see e.g.[MN, Prop. 4.5]). It is also easy to show that there is an alternative but equivalent definition of the contact group, as the set of diffeomorphisms of $(\mathbb{R}^n \times \mathbb{R}^p, (0,0))$ of the form $(x, y) \mapsto (h(x), \varphi(x) \cdot y)$, where $\varphi : (\mathbb{R}^n, 0) \to \mathrm{Gl}_p(\mathbb{R})$ is a smooth map-germ.

A map-germ (f, x_0) is $\mathcal{K}$-*simple* if there is a neighborhood W of a representative of (f, x_0) and a neighborhood of x_0 for which there are only finitely many $\mathcal{K}$-equivalence types among the germs (g, x), with $g \in W$ and $x \in U$.

To understand Jim Damon's contribution to singularity theory it is helpful to remind ourselves of the state of the subject when he entered it. (As a warning, this sketch is brief, and for additional brevity we have been rather loose with some of our definitions, descriptions and statements.)

Whitney had completed some characteristically pioneering work on stable mappings from the plane to itself, published in 1955, after some fundamental papers on immersions and embeddings much earlier (see [Wh1], [Wh2], [Wh3]). Whitney also extended his 1955 paper to more dimensions in [Wh5]. Building on Whitney's results, René Thom (see [T1], [T2], [T3], [T4], [T5], [T6], [T7]) then sketched out a general theory of stable mappings.

Thom had three initial insights: the first was that one can locally model singularities in finite dimensional jet spaces, that is spaces of polynomial maps of a given degree. The second was that the natural mappings into the resulting jet-bundles did not have problematic integrability conditions and hence he could prove "Thom's transversality theorem" which apparently Whitney did not at first believe. The third was to consider infinitesimal versions of stability, seeking diffeomorphisms by integrating vector fields.

He showed that smoothly stable mappings are not dense, indeed for some pairs of manifolds did not exist, and made the major step of introducing a local version of right-left stability, "homotopic" stability: roughly speaking, a germ of map $f : (N, S) \to (P, y_0)$ is homotopically stable if for every deformation f_u, depending smoothly on the parameter $u \in \mathbb{R}^d$, there exist families of diffeomorphisms ψ_u and φ_u, also depending smoothly on u, such that $f_u = \psi_u \circ f \circ \varphi_u$. Thom also understood the potential value of having a smooth (C^∞) version of the Weierstrass preparation theorem from complex analysis, which he apparently persuaded/nagged Malgrange into proving! The smooth preparation theorem became one of the principal technical tools in Thom's and Mather's later work on smooth stability. Finally, Thom produced a series of increasingly mysterious papers which provided important steps towards proving his conjecture that topologically stable maps between any two manifolds were dense. This was a remarkable, if technically incomplete, corpus of work, perhaps not as appreciated as it should be. In retrospect the profound can look obvious. That said, Thom's project had very significant gaps – most of which were filled by John Mather.

Before continuing, we introduce some more notions.

- For a smooth mapping f from an open set U in $\mathbb{R}^n$ into $\mathbb{R}^p$, and k a non-negative integer, the *k-jet* of f at $x \in U$ is the tuple of derivatives up to order k of f at x: $(x, f(x), Df(x), \ldots, D^k(f(x))$. The space of k-jets $J^k(U, \mathbb{R}^p)$ is an open set in a Euclidean space, projecting to U. These notions extend to the case of maps between manifolds, $f : N \to P$, and the map $j^k f$ from N to $J^k(N, P)$ is a smooth map. Thom's Transversality Theorem says that if X is a submanifold of the k-jet bundle $J^k(N, P)$, then the set $T(X)$ of maps f for which $j^k f$ intersects X transversally is a countable intersection of open, dense sets, and so is dense; if X is closed, then $T(X)$ is actually open as well as dense. Note that the jet extension maps $j^k f$ are far from general mappings: most mappings $N \to J^k(N, P)$ are not jet-extension maps, the latter satisfy the "integrability conditions" mentioned above. Presumably this is why Whitney found it hard to believe Thom's result, which in fact is not very difficult to prove.
- A map f is said to be *k-determined* (or $\mathcal{A}$-k-determined) at x if, for every g such that $j^k g(x) = j^k f(x)$, g is $\mathcal{A}$-equivalent to f in some neighborhood of x, and *finitely determined* is it is k-determined for some finite k. The Implicit Function Theorem implies that any mapping f with derivative $Df(x)$ non-singular is 1-determined at x, and a theorem of Morse (the "Morse Lemma") shows that a real-valued function f with $Df(x) = 0$ but $D^2 f(x)$ non-degenerate is 2-determined at x. A mapping f is *finitely determined* (or $\mathcal{A}$-finite) at x if it is k-determined at x for some finite k. One similarly defines $\mathcal{R}$-finiteness and $\mathcal{K}$-finiteness.
- An *unfolding of a map* $f : U \to \mathbb{R}^p$ is a map F from $U \times \mathbb{R}^s$ to $\mathbb{R}^p \times \mathbb{R}^s$ of the form $F(x, u) = (F_1(x, u), u)$. For example, $f(x) = x^3$ is not stable, and $F(x, u) = (x^3 + xu, u)$ is stable, so F is a *stable unfolding* of f.

Enter John Mather, Jim's PhD supervisor. In a series of astonishing papers (see [M1],[M2],[M3],[M4],[M5],[M6],[M7]) published between 1968 and 1971 Mather set out an enormous generalisation of the work of Whitney and Thom, all the more amazing since much was the content of his Ph.D thesis, written under the guidance of John Milnor. Building on, clarifying and refining Thom's ideas, as well as introducing a significant number of new core concepts, Mather showed that the density of stable mappings between manifolds only depended on the respective dimensions of the source and target, (n, p), determined for which pairs density held, the "nice dimensions", and gave a method for determining the local structure of stable mappings. One of Mather's new ideas was the notion of multi-jet spaces ${}_rJ^k(N, P)$, multi-jet extensions ${}_rj^k f : N^{(r)} \to_r J^k(N, P)$ and multi-transversality, introduced to deal with the issue raised above of self-intersections in the image of a map.

He also developed a broad theory of unfoldings and finite determinacy and proved Thom's conjecture on topological stability which we discuss below.

A central ingredient in Mather's work on smooth stability was the following theorem.

Theorem 1.1.1 *The following are equivalent for a smooth proper map $f : N \to P$: For $k \geq p$ and $r \geq p+1$,*

1. *f is smoothly stable;*
2. *${}_rj^kf$ is transverse to all $\mathcal{A}$-orbits in the multi-jet space ${}_rJ^k(N,P)$;*
3. *${}_rj^kf$ is transverse to all $\mathcal{K}$-orbits in the multi-jet space ${}_rJ^k(N,P)$*

The relations of stability to transversality is what made possible the use of Thom and Mather's transversality theorems to prove the density of the set of stable mappings in the space of proper mappings, in the nice dimensions, and the density of topologically stable maps in the space of proper mappings in all dimensions.

Unusually in the development of a major subject, having created the core of modern singularity theory, Mather almost immediately moved away to do deep work in foliations, Hamiltonian dynamical systems and other areas. (This proved very frustrating for Jim, who could not persuade Mather to complete their joint book on the stability of mappings started in the early 1970's!) At around the same time Arnold started his work in the subject principally focusing on functions (target dimension one), building on earlier work of Mumford, Brieskorn and Milnor and introducing his theories of Lagrange and Legendre singularities, closely related to Thom's sometimes controversial catastrophe theory.

This was an exciting time, and with Mather's unexpected departure from the scene there was a short pregnant pause. There were lots of questions to be answered and avenues to be explored. In the broadest of terms we set some of them down and then reflect on how Jim's seminal papers over a period of 50 years have helped transform our subject. It should be noted at the outset that Mather's work was complete in one sense: he completely solved the question of the relation between the local (homotopic) stability of the germs of a map $f : N \to P$ and the global stability of f. For this reason, outside the area of topological stability, most subsequent developments (Jim's included) have focused on germs of maps rather than on (global) maps between manifolds.

1. Mather's proof that topologically stable mappings were dense was not constructive; indeed, until the mid-70s there was not a single example of a topologically stable mapping that was not smoothly stable! What could be said about them? It was clear that smoothly stable mappings were topologically stable. Was the converse true in the nice dimensions and how could one find mappings which were topologically but not smoothly stable?

2. It was immediately obvious that Thom and Mather's ideas applied to other situations: for example, those where the manifolds had a group acting, where the diffeomorphisms are required to preserve a submanifold or some other subset of source or target. There was a period when many of us working

in singularity theory adapted and amended Mather's results on determinacy and unfoldings to new settings, often arising from interesting geometric applications. But it was all rather chaotic. What assumptions were needed to establish the sort of results these applications required? And what were the analogous results and geometry?

3. The theory of functions (Arnold, his antecedents, his students and many others) produced some astonishingly beautiful mathematics. Local monodromy provided links to braid groups, reflection groups, triangle groups, Dynkin diagrams and Hodge theory. The study of liftable and lowerable vector fields tangent to stable Legendre and (augmented=full bifurcation sets) Lagrange singularities showed they were free divisors – that is, the module of ambient vector fields which are tangent to them is locally free. There were beautiful applications to differential geometry and optics. By comparison the theory of smooth mappings when the target dimension was ≥ 2 seemed a little dull. While some of the work for functions had been generalised to complete intersections, what could take the place of the Milnor fibration when looking at mappings up to $\mathcal{A}$-equivalence? And how much geometry could be extracted from the corresponding spaces of mappings? (There was also the obvious question of finding efficient criteria for finite determinacy and listing mappings, as Arnold had for functions, but this was one of the few areas that Jim did not address. He was busy with other things!)

4. There were also many natural, interesting highly singular objects that seemed to share some of the properties of isolated hypersurface singularities. One question, rather like an amalgam of (2) and (3) was to find a set of properties enjoyed by these objects which allowed these generalisations of Milnor fibre, Milnor number and higher multiplicities.

5. Finally, singularity theory is, after all, what became of one of the greatest breakthroughs in human intellectual endeavour, the differential calculus. So, we could ask, like calculus, what were the interesting applications to geometry or physics of these new revelations?

Jim Damon made profound contributions in each of these areas.

1.1.1 Topological stability

As mentioned above, Thom had shown with an example that smoothly stable maps $N \to P$ are not necessarily dense in the space of maps $C^\infty(N, P)$ with its Whitney topology. The failure of density is somewhat subtle; it occurs when there are families of mappings whose smooth equivalence class varies continuously. In such a family, any one of its members can be deformed an arbitrarily small amount to an inequivalent map, and is thus not stable. The map from four copies of $\mathbb{R}$ to the plane, consisting of four distinct immersed straight lines all passing through 0, is such a family: the cross ratio of the four tangent lines is an isomorphism invariant, so, for instance, rotating the image of one of the immersions about 0 while keeping the other three fixed produces a smoothly inequivalent map. Although a straightforward consequence of the fact that diffeomorphisms are by definition locally linearisable, it is not something one can 'see'! On the other hand, all such maps are *topologically* equivalent to one another: it is possible to transform any four concurrent lines to any other four by means of a homeomorphism (which is not differentiable at 0). This example is not topologically stable, of course: the quadruple intersection can easily be removed, e.g. by moving one of the lines so that it does not pass through 0. But in more complicated situation, where a germ or multigerm which belongs to a continuum of smooth equivalence classes is stably present in a map, then every neighbourhood of the map contains maps inequivalent to it, so that there is no stable map nearby.

In 1970 John Mather established that topologically stable mappings are dense in the space of smooth mappings between any two manifolds. The proofs built on work of Thom but again with vital new ingredients from Mather. The initial part was published in booklet form with more detail in two papers on stratifications and mappings, the key publication being in 1976 (see [M7], [M8], [M9]). An alternative account was given in a book by Gibson, Wirthmüller, du Plessis and Looijenga in the same year (see [GWPL]), with a particularly significant contribution of the final author following his work on the topological stability of smooth families of mappings. Both proofs were effectively non-constructive, depending on the existence of Whitney stratified sets, for whose construction no practicable algorithm was provided.

A *stratification* of a subset A of a manifold M is a locally finite partition of A into submanifolds of M, called *strata*, satisfying the condition of the frontier: if a d-dimensional stratum S intersects the closure of a d'-dimensional stratum S', then S is contained in the closure of S'. It says something for the genius respectively of Whitney and Thom that they recognised how to express the notion that stratifications "fit together nicely", and exploited that to establish results about topological stability. We start by supposing we have a stratification of an open subset of $\mathbb{R}^N$. The stratification satisfies *Whitney's condition (A)* if, for every pair of strata S', S, whenever $x_i \in S' \to y \in S$, and the tangent spaces $T_{x_i}S'$ approach a limit d'-plane T, then T contains T_yS. The stratification satisfies *Whitney's condition (B)* if for every pair of strata S', S, for each sequence x_i of points in S' and each sequence y_i of points in S, both converging to the same point $y \in S$, such that the sequence of secant lines L_i between x_i and y_i converges to a line L, and the sequence of tangent d' planes T_{x_i} converges to a d'-plane T, then L is contained in T. Condition (B) implies condition (A). A *Whitney stratification* is one which satisfies condition (B). One easily checks that these local notions are invariant under diffeomorphism, so there is a corresponding notion of Whitney (A) or (B) regularity for any stratification of a smooth manifold. (To sharpen their understanding the reader might like to consider the so-called Whitney umbrella $x^2 = zy^2$ in $\mathbb{R}^3$ in Example 1.1.11.) Whitney showed in [Wh4] that every analytic set possesses a Whitney stratification, and Thom showed in

[T6] that (B) implies that the stratification is topologically trivial along each stratum. For more information about Stratification Theory see [Tr].

In [M7], [M8] and [GWPL] Mather showed that for any two manifolds N, P, there is a positive integer k and a stratification $\mathcal{S}$ of the jet-space $J^k(N,P)$ (his 'canonical stratification') with the property that any proper map $f : N \to P$ which is multi-transverse to a natural stratification of some ${}_rJ^k(N,P)$ constructed from $\mathcal{S}$ is topologically stable. Moreover the choice of k, r and the construction of $\mathcal{S}$ depends only on the dimensions of N and P. So by Mather's multi-transversality version of the Thom transversality theorem, the set of maps satisfying the transversality conditions is dense, and hence topologically stable maps are dense.

Despite a lot of work undertaken in Liverpool, at the insistence of Terry Wall, few natural non-trivial examples were known beyond those concerning families of hypersurface singularities deducible from Teissier's famous work on vanishing cycles, plane sections and Whitney conditions. Equally problematic, it turned out that the sufficient condition for topological triviality was not necessary. Indeed in 1975 Looijenga wrote a break-through paper, [ELII], establishing the topological triviality of the discriminants of the simple elliptic singularities along the modulus, while in 1978 Bruce showed, in [Br], that although Whitney (A)-regularity held along the modulus, Whitney (B) regularity failed for $\tilde{E}_6$ for certain values, though this could be compensated for by its weighted homogeneity. Indeed, the first explicit non-trivial stratum from the Thom-Mather methodology was only obtained by Wall in 1980.

There were two key questions:

(I) In the nice dimensions does topological stability coincide with smooth stability? The question has two halves:

(a) Is it possible for two smoothly inequivalent stable mappings in the nice dimensions to be topologically equivalent?

(b) Could it be true that every topologically stable mapping in the nice dimensions is in fact smoothly stable?

(II) Outside the nice dimensions (dimension by dimension) what are the topologically stable mappings which are not smoothly stable?

Naturally in moving to topological equivalence one loses all the familiar tools of singularity theory: local intersection numbers and indeed transversality are not invariant under homeomorphism, and there is no infinitesimal theory, and hence none of the quite sophisticated algebra which appears in the smooth theory. Even more curiously, there is not even a *group* of equivalence acting on the space of maps. While it may happen that non-smooth homeomorphisms Ψ and Φ transform a given smooth f to a smooth g, the same pair of homeomorphisms will in general transform a smooth h to something not smooth.

Problem I was first considered in the thesis of Robert May written under John Mather's supervision (see [May]), where he noted some interesting pathologies in the non-compact case, introduced the notions of topological transversality and uniform stability, and established (I(a)) when $n > p, p < 7, n < 2(n - p + 2)$, for uniformly stable maps. With this as his starting point, Jim first proved

Theorem 1.1.2 (Paper 11) *In the nice dimensions, the topological and smooth classifications of smoothly stable germs agree.*

This followed from the equally striking fact, which Jim also proved, that for $\mathcal{K}$-simple germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0),\ n \leq p$, the isomorphism type of the complexification $\mathcal{Q}(f) \otimes_{\mathbb{R}} \mathbb{C}$ of the local algebra $\mathcal{Q}(f)$ is a topological invariant. Mather had proved that smoothly stable germs are smoothly equivalent if and only if their local algebras are isomorphic.

For his second result we need a new notion (related to May's "uniform stability"). A C^0-stable mapping $f : N \to P$ is S*-stable* if $f \times I : N \times T^k \to P \times T^k$ is C^0-stable for all k-tori $T^k, k \geq 0$. Note that T^k is chosen as an extra factor merely because it is one of the simplest compact manifolds of dimension k. The definition of S-stability highlights another surprising difference between smooth stability and topological stability. If f is smoothly stable then $f \times I$ is also smoothly stable. This does not seem obvious, but is relatively easy to show: essentially, smooth stability of f is equivalent to the transversality of $j^k f$ to all $\mathcal{K}$-orbits in $J^k(N, P)$ (plus an analogous statement for multi-jets); the conclusion follows from the fact that the inclusion $J^k(N, P) \to J^k(N \times S, P \times S)$ sending any k-jet $j^k g(x_0)$ to $j^k(g \times 1)(x_0, s_0)$ (for any manifold S and point $s_0 \in S$), is transverse to all $\mathcal{K}$-orbits. The analogous statement is not true for topological stability (which helps to show that the statement for smooth stability is not "obvious"!). One can appreciate this using an example due to Mather's student May: the drawing below shows the graph of a function $f : \mathbb{R} \to \mathbb{R}$ with infinitely many non-degenerate maxima and minima, all with different critical values, and a degenerate critical point at (0,0).

In the region of 0, f is equivalent to $x \mapsto -x^3$, which is not a topologically stable germ, but even so, globally f is topologically stable: if it is perturbed in the region of 0 so that a new maximum and minimum appear, then since there are already an infinite number of maxima and minima, and since in a neighbourhood of 0, f is topologically equivalent to the immersion $x \mapsto -x$, qualitatively the picture looks the same: topologically the new map is equivalent to the old. Notice, though, that the homeomorphisms in source and target transforming the new map into the old cannot be close to the identity. If a new minimum and maximum appear in the neighbourhood of 0, the compensating homeomorphism of the domain has to shift them either to the critical points at 1 and 2.5 or to the critical points at -2 and -1, and in consequence relocate all the other critical points too. Notice also that of course f is not smoothly stable: the degenerate critical point can be removed by an arbitrarily small perturbation, and degenerate critical points are detected by smooth equivalence.

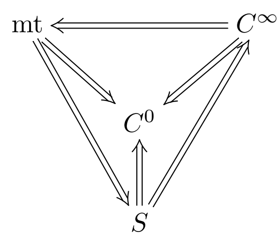
However, $f \times I : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ is not topologically stable – it is possible to deform it in a neighbourhood of $(0, t)$, for any t, via the so-called "lips" bifurcation, so that a circle of critical points appears, in which all except two of are fold points and the others are Whitney cusps. In $f \times 1$, all of the fold points lie on straight lines and there are no Whitney cusps, so the deformed map is not topologically equivalent to $f \times I$.

Jim's theorem in Paper 13 is then:

Theorem 1.1.3 *In the nice dimensions, if N is compact and $f : N \to P$ is S-stable then it is smoothly stable.*

This is not quite a full answer to (I) but close to it.

Jim's survey in Paper 10 is a good introduction to the subject, and covers most of what was known at the time. His Paper 13 illustrates the relationships between the various versions of stability in the nice dimensions with the following diagram:



Here "mt" is transversality to Mather's canonical stratification of jet space, shown by Mather to be a sufficient, but not necessary condition for topological stability, and "S" is S-stability.

There are other versions of topological stability, and a great deal of work continues to be done, relating the various notions. The recent survey by Maria Ruas [Ru] gives a good account of the current state of play, and makes clear the lasting influence of Jim's work.

We now turn to problem (II). As already mentioned, Looijenga had produced a new approach to topological triviality in his paper on simple elliptic singularities. This depended on the fact that the unfoldings were naturally weighted homogenous. The proof was rapidly generalised by Wirthmüller to unfoldings of hypersurface singularities along the Hessian direction, but it was Jim who first realised that Looijenga's construction was intimately linked to the notion of $\mathcal{A}$-finite determinacy. His result concerns unfoldings F of non-negative weight of a weighted-homogeneous polynomial germ f. If $f(x) = (f_1(x), \ldots, f_p(x))$ is weighted homogeneous, with $\mathrm{wt}(x_i) = a_i > 0, i = 1, \ldots, j$ and $\mathrm{wt}(f_j) = d_j, j = 1, \ldots, p$, then a polynomial unfolding

$$F(x, u) = (f_1(x) + \sum_{\alpha_1} u_{\alpha_1} x^{\alpha_1}, \ldots, f_p(x) + \sum_{\alpha_p} u_{\alpha_p} x^{\alpha_p}, u),$$

where the u_{α_j} are functions of the parameters u_i, is *of non-negative weight* if whenever $u_{\alpha_j} \neq 0$ then $\mathrm{wt}(x^{\alpha_j}) \geq d_j$.

Theorem 1.1.4 (Papers 17 and 18) *If f is a finitely $\mathcal{A}$-determined germ, then any polynomial unfolding of F of non-negative weight is topologically trivial.*

Jim also extended this to the case where some of the variables have non-positive weight. Identifying the key ingredients in Looijenga's proof, he extended its scope hugely. His theorem applies to mappings as well as functions, which included in particular unimodal singularities defined by complete intersections, and finitely determined singularities which are not complete intersections.

As one might expect both aspects of this work were extremely influential, for example in the project undertaken by du Plessis and Wall on topological stability (see [dPW]).

1.1.2 Subgroups of Mather's groups

In what follows, almost all of the theory applies both to smooth maps between real manifolds, and analytic maps between complex manifolds. So we will use "$\mathbb{K}^n$" to denote either (or both) of $\mathbb{R}^n$ and $\mathbb{C}^n$, and "smooth" to mean either C^∞ in the real case and analytic in the complex case. Likewise, "diffeomorphism" will mean "smooth diffeomorphism" in the real case and "biholomorphic diffeomorphism" in the complex case. When considering complex analytic germs, the ring $\mathcal{E}_n$ of germs at 0 of smooth functions should be replaced by the ring $\mathcal{O}_n$ of germs at 0 of complex analytic functions. Note that $\mathcal{O}_n$ is Noetherian while $\mathcal{E}_n$ is not. (This, along with the failure over $\mathbb{R}$ of the Nullstellensatz, is one of the many reasons why complex analytic geometry is more popular than real smooth geometry.) Where we refer to germs $\mathbb{K}^n \to \mathbb{K}^p$, $\mathcal{E}_n$ should be understood as denoting either $\mathcal{E}_n$ in the real case or $\mathcal{O}_n$ in the complex case

Mather's groups are the global and/or local diffeomorphism groups $\mathcal{A}$, $\mathcal{K}$, and their subgroups $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{C}$, where $\mathcal{C}$ consists of those contact diffeomorphisms which fix $\mathbb{K}^n$ pointwise. The main theorems Mather proved for $\mathcal{A}$ are the finite determinacy theorem, the versal unfolding theorem, and the "classification theorem" (that stable germs are classified by their contact class). Let $f : \mathbb{K}^n, 0 \to \mathbb{K}^p, 0$ be a smooth mapgerm and $\mathcal{G}$ a group acting on the space of such map-germs, such as the above Mather groups.

Suppose $f_t : N \times I \to P$ is a smooth family of maps, $I \subseteq \mathbb{K}$ is an interval, and $f_0(x) = f(x)$. For each x, $\phi_x(t) := f_t(x)$ is a smooth curve in P so $\phi_x'(0)$ is a tangent vector to P at $f(x)$. Varying x, we get a "vector field along f"; the set of all these is denoted $\theta(f)$. As a special case, if $f : N \to N$ is the identity map (so that for t near zero, each f_t is a diffeomorphism) then the "vector field along f" is simply a vector field, in the usual sense, on N. Conversely any vector field integrates to give a 1-parameter family of diffeomorphisms $f_t(X)$ with $f_0(x) = x$. Thus, the set of smooth vector fields on N can be thought of as the tangent space, at the identity map, to the group Diff(N). The spaces of vector fields on N and P are denoted $\theta(N)$ and $\theta(P)$, respectively. For families of maps and diffeomorphisms F, H on N and K on P with $F(x,0) = f(x)$, $H(x,0) = x$ and $K(y,0) = y$, let $G = K \circ F \circ H$, so $G(x,0) = f(x)$. Differentiating G with respect to t and setting $t = 0$ yields $\alpha \in \theta(N)$ and $\beta \in \theta(P)$ and, by the chain and product rules, $df \cdot \alpha + \beta \circ f \in \theta(f)$, which we can consider a tangent vector at f to the $\mathcal{A}$ orbit of f. So we define the tangent space of this orbit to be

$$T_f\mathcal{A} \cdot f = tf(\theta(N)) + \omega f(\theta(P)),$$

where $tf(\alpha) = df(\alpha)$ and $\omega f(\beta) = \beta \circ f$. We say f is *infinitesimally stable* if the tangent space at f to the $\mathcal{A}$-orbit of f equals the tangent space at f to $\mathcal{F}$, i.e.

$$\theta(f) = tf(\theta(N)) + \omega f(\theta(P)).$$

Mather showed in [M2] that for proper smooth maps $N \to P$, infinitesimal stability is equivalent to stability.

Now consider the local case. We denote by θ_n the space of germs at 0 of vector fields on $\mathbb{K}^n$, and $\theta(f)$ the space of germs at 0 of vector fields along f. A vector field has n components, so θ_n can be identified with $\mathcal{E}_n^n$, and similarly $\theta(f)$ can be identified with $\mathcal{E}_n^p$. Let $\mathcal{F}$ be the set of all map-germs $(\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. Since we are requiring $0 \in \mathbb{K}^n$ to map to $0 \in \mathbb{K}^p$, $\mathcal{F}$ can be identified with $\mathfrak{m}_n\mathcal{E}_n^p$, where $\mathfrak{m}_n$ is the maximal ideal in $\mathcal{E}_n$, consisting of all smooth germs which vanish at 0. The group of right-left equivalences acting on $\mathcal{F}$ consists of pairs of diffeomorphisms $\phi \in \text{Diff}(\mathbb{K}^n, 0)$ and $\psi \in \text{Diff}(\mathbb{K}^p, 0)$. The vector fields generating diffeomorphisms fixing 0 must vanish at 0, so the tangent space to this group is $\mathfrak{m}_n\theta_n \times \mathfrak{m}_p\theta_p$. Fix $f \in \mathcal{F}$. Composition with f and differentiation, as in the global case, yields a tangent space to the $\mathcal{A}$ orbit of f in $\mathcal{F}$:

$$tf(\mathfrak{m}_n\theta_n) + \omega f(\mathfrak{m}_p\theta_p) = \mathfrak{m}_n df \mathcal{E}_n^n + f^*(\mathfrak{m}_p^p).$$

Similarly the tangent space to the $\mathcal{K}$ orbit of f at f is identified with

$$tf(\mathfrak{m}_n\theta_n) + f^*\mathfrak{m}_p\theta(f).$$

The $\mathcal{G}$-*codimension* of f for any of these groups is the vector space codimension of the orbit tangent space in the tangent space to $\mathcal{F}$, $\mathfrak{m}_n\theta(f)$. (A good reference for this material is the survey paper [Wa].)

If we allow the source and target basepoints to vary we get the *extended tangent spaces*

$$T\mathcal{A}_e \cdot f = tf(\theta_n) + \omega f(\theta_p) = df(\mathcal{E}_n^n) + f^*\mathcal{E}_p^p$$

and

$$T\mathcal{K}_e \cdot f = tf(\theta_n) + f^*\mathfrak{m}_p\theta(f) = df(\mathcal{E}_n^n) + (f_1, \ldots, f_p)\mathcal{E}_n^p.$$

This takes us out of the context of a group action, but is necessary in order to study the stability of germs (by which we mean homotopic stability). For this, we need to allow for the possibility that the interesting behaviour of the germ changes its location, while remaining essentially unchanged. To transform such a deformation f_u of f back to f, it may be necessary to move the origin. Consider the germ $f(x) = x^2$. In the unfolding $F(x,u) = (x^2 + ux, u)$, the critical point at 0 moves to $-u/2$, and the critical value moves to $-u^2/4$. The unfolding is trivialised by families $\Phi(x,u) = (x - u/2, u)$ and $\Psi(y,u) = (y + u^2/4, u)$ – that is, $\Psi \circ F \circ \Phi(x,u) = (x^2, u)$. The behaviour of f at the critical point has not changed. Indeed, f is stable. In [M2], Mather proved the local version of the theorem that infinitesimal stability is equivalent to stability: a germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is stable if and only if

$$tf(\theta_n) + \omega f(\theta_p) = \theta(f).$$

In [M3], Mather proved the following "Finite Determinacy Theorem" for $\mathcal{G} = \mathcal{A}, \mathcal{K}$ and $\mathcal{R}$.

Theorem 1.1.5 *If f has finite $\mathcal{G}$-codimension then it is finitely determined for $\mathcal{G}$-equivalence, and conversely.*

Mather's estimates for the $\mathcal{A}$-determinacy degree were greatly improved by the pioneering work of Gaffney, then du Plessis ([G1, G2], [dP] and [GdP]) and, later, by Bruce, du Plessis and Wall in [BdPW].

The unfolding

$$F(x,u) = (f(x,u), u) : (\mathbb{K}^{n+q}, 0 \to (\mathbb{K}^{p+q}, 0)$$

of $f_0(x)$ is $\mathcal{G}$-*versal*, if for any other unfolding

$$G(x,v) = (g(x,v), v) : (\mathbb{K}^{n+r}, 0 \to \mathbb{K}^{n+r}, 0)$$

of $f_0(x)$ there is a germ

$$\lambda : (\mathbb{K}^r, 0) \to (\mathbb{K}^q, 0)$$

so that $F(x, \lambda(v))$ is $\mathcal{G}$-equivalent to G as an unfolding via an unfolding equivalence $\psi(x,v) = (\psi_1(x,v), v)$ with $\psi_1(x,0) = x$. The following "Versal Unfolding Theorem" was proved for $\mathcal{G} = \mathcal{R}$ by Mather, and for $\mathcal{G} = \mathcal{A}$ and $\mathcal{K}$ by Jean Martinet in [Mt].

Theorem 1.1.6 *(Versal Unfolding Theorem). For the above groups $\mathcal{G}$, the unfolding $F(x,u)$ of $f_0(x)$ on q parameters $u = (u_1, \ldots, u_q)$ is a $\mathcal{G}$-versal unfolding iff*

$$T\mathcal{G}_e \cdot f_0 + \langle \frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_q} \rangle = \theta(f_0). \tag{1.1}$$

For example let $f_0(x) = x^4$ and $\mathcal{G} = \mathcal{A}$. Then $\theta(f_0) = \mathcal{E}_1$ and the extended tangent space of $\mathcal{A} \cdot f_0$ is

$$f_0' \cdot \mathcal{E}_1 + f^*(\mathcal{E}_1) = 4x^3\mathcal{E}_1 + \langle 1, x^4, x^8, \ldots \rangle$$

Thus $F(x, u, v) = (x^4 + ux + vx^2, u, v)$ is a versal unfolding.

These two theorems are crucial for the classification of stable and finite-codimension germs of mappings, as well as many of the appications of singularity theory to differential geometry; see later sections.

It was clear from the outset that the Thom-Mather approach to determinacy and unfoldings applied to a wide range of situations. A whole host of examples naturally arose where Mather's theorems did not apply but his infinitesimal methods clearly could at least be hoped to apply. These included the case of diagrams of mappings, mappings invariant under group actions, mappings with a distinguished subvariety of the source or target (which must be preserved by the diffeomorphisms used to define "equivalence"), bifurcation theory, distinguished parameters etc. Each application involved a new set of definitions and proofs. As early as 1977 Jim was thinking about a natural set of conditions on a linear space of mappings $\mathcal{F}$ and a group of diffeomorphisms $\mathcal{G}$ acting on $\mathcal{F}$ which would mean that one could apply the Mather 'yoga'. The result was a very natural definition of "geometric subgroups" of $\mathcal{A}, \mathcal{K}$ (as one might guess involving some challenging algebra) which encompassed more or less all known examples and for which the results corresponding to 1.1.5 and 1.1.6 held.

We do not give the definition of geometric subgroup here, since it requires considerable explanation. However, the following are examples.

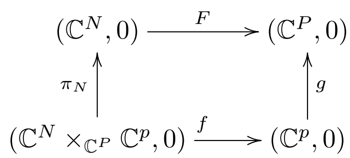
Example 1.1.7 1. In [Ar] V.I. Arnold introduced the subgroup $\mathcal{R}_\partial$ of $\mathcal{R} = \mathrm{Diff}(\mathbb{K}^n, 0)$, consisting of germs of diffeomorphisms of $(\mathbb{K}^n, 0)$ mapping the hypersurface $\mathbb{K}^{n-1} \times \{0\}$ to itself, and classified germs of functions on $(\mathbb{K}^n, 0)$ under this group, referring to the equivalence classes as *boundary singularities*. In particular, he found the list of $\mathcal{R}_\partial$–simple boundary singularities, and uncovered remarkable links between the singularities in the list and certain simple Lie groups, extending earlier work of Brieskorn ([Bri]) and other—see [Du] for an overview.

2. The subgroup group $\mathcal{K}_V$ of $\mathcal{K}$, acting on the space of germs $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, where V is a subvariety of $(\mathbb{K}^p, 0)$, and the diffeomorphism Φ of $(\mathbb{K}^n \times \mathbb{K}^p, (0, 0))$ is required to map V to itself, though not necessarily fixing V pointwise. In the case that V is a hypersurface, Jim used this to great effect with his work on almost free divisors, discussed below. In Paper 35 he showed that the extended $\mathcal{K}_V$ tangent space to g is equal to $tg(\theta_n) + g^*\mathrm{Der}(-\log V)$, where $\mathrm{Der}(-\log V)$ is the $\mathcal{O}_p$-module of ambient vector fields which are tangent to V at its smooth points. Moreover, in the case that V

is the discriminant of a stable map-germ $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^P, 0)$ and g is "$\mathcal{K}_V$-finite" and transverse to F, then

$$\frac{\theta(g)}{T\mathcal{K}_{V,e}g} \simeq \frac{\theta(f)}{T\mathcal{A}_e f} \tag{1.2}$$

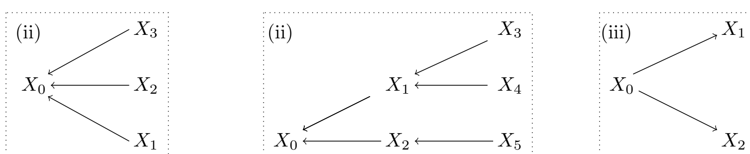
where f is projection to $(\mathbb{C}^p, 0)$ of the fibre product $(\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p, 0)$, as in the following pull-back diagram:



Note that the fibre product is non-singular because g is transverse to F.

It follows from (1.2) that an unfolding G of g induces an $\mathcal{A}_e$-versal unfolding F of f if and only if G is $\mathcal{K}_V$ versal.

3. Spaces of convergent diagrams of mappings, as in diagrams (i) and (ii) below, with the group of diffeomorphisms $\prod_i \mathrm{Diff}(X_i, 0)$ acting in the obvious way. Note that diagrams like (i), where no compositions are involved, are simply multi-germs, as described at the start of Section 1.1. The same group, acting on spaces of divergent diagrams of mappings, like (iii), is *not* a geometric group in Jim's sense.



4. Let G be a subgroup of $\mathrm{Gl}_p(\mathbb{K})$, acting in the standard representation on $\mathbb{K}^p$. We consider the subgroup of $\mathcal{K}$ consisting of diffeomorphisms H of $(\mathbb{K}^n \times \mathbb{K}^p, (0,0))$ of the form $(x, y) \mapsto (h(x), \varphi(x) \cdot y)$, where $\varphi : (\mathbb{K}^n, 0) \to G$ is a smooth map. When $G = \mathrm{Gl}_p(\mathbb{K})$, this gives the usual version of $\mathcal{K}$-equivalence, as remarked above. The resulting equivalence relation, with $G = \mathrm{Gl}_p \times \mathrm{Gl}_q$, acting by multiplying matrices on the left and right, has been used to classify n-parameter families of $p \times q$ matrices by Bruce, Tari, Fruhbis-Kruger, Goryunov, Zakalyukin (e.g. in [Br2, BruTa, GoZa, GoMo, Fru]) and others, and in many other contexts, e.g. by Izumiya *et al* ([Iz]) .

Theorem 1.1.8 (Papers 20 and 24) *If $\mathcal{G}$ acting on $\mathcal{F}$ is a geometric subgroup, and if $f \in \mathcal{F}$ has an orbit $\mathcal{G}.f$ of finite codimension in $\mathcal{F}$ then*

(a) the linear versality criterion (1.1) *holds, and*

(b) f is finitely determined for $\mathcal{G}$-equivalence.

These results have been used time and again by researchers in singularity theory, with the effect of both unifying and enriching the examples that enable our understanding of our attractively diffuse subject.

In Paper 30, Jim also extended his work on topological triviality to show that the analogs of 1.1.5, 1.1.6 and 1.1.8 also hold.

1.1.3 Moving on from isolated hypersurface singularities

As mentioned in the introduction there was an explosion of work on isolated hypersurface singularities in the 1960s and early 70s: Milnor fibrations, monodromy groups, mixed Hodge structures, higher multiplicities, equisingularity results, geometric interpretations of discriminant and bifurcation varieties. It was natural to seek generalisations to the case of mappings. One particular challenge was to find an analogue for the Minor fibration for $\mathcal{A}$-finite mappings $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, after earlier work of David Mond [Mo] in the case $n = 2, p = 3$.

The most natural answer seemed to be to consider the "fibration" over the complement of the bifurcation set in the base U of a suitable representative of a versal unfolding $(x, u) \mapsto f_u(x)$, whose 'fibre' over $u \in U$ is the mapping f_u (which we will refer to as a 'stable perturbation' of $f = f_0$). After all, f_u is the "nearby stable mapping" associated to the unstable map-germ f, in the same way that the Milnor fibre of a hypersurface singularity is the "nearby smooth space" associated to the singular space germ $\{f = 0\}$. But how to derive numerical invariants from this fibration? Mond had sought invariants in the image of f_u in the case $n = 2, p = 3$; Jim Damon suggested that when $n \geq p$, one should instead look at the discriminant – the set of critical values. After all, for $\mathcal{A}$-finite germs with $n \geq p - 1$, this too is a hypersurface in $\mathbb{C}^p$. This follows from the fact that because f has isolated instability, $j^1 f$ is transverse, outside 0, to every $\mathcal{A}$-invariant submanifold of the jet space $J^1(n, p)$ (which is just the set of linear maps $\mathbb{K}^n \to \mathbb{K}^p$), and in particular to the set of singular linear maps. Thus, the codimension in $\mathbb{K}^n$ of the critical set of f, Σ_f, is equal to the codimension of the set of singular linear maps in the space of linear maps $\mathbb{K}^n \to \mathbb{K}^p$. This codimension is $n - p + 1$, so that the dimension of Σ_f is $p - 1$. An $\mathcal{A}$-finite germ is finite-to-one on its critical locus, so the image under f of Σ_f has the same dimension. Following Jim's idea, he and Mond proved the following theorem.

Theorem 1.1.9 (Paper 41) *Suppose that $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is finitely $\mathcal{A}$-determined with $n \geq p$. Then the discriminant of a stable perturbation of f has the homotopy type of a wedge of spheres of dimension $p - 1$. Denote the number of spheres in the wedge by $\mu_\Delta(f)$ – it is the "discriminant Milnor number". If (n, p) is in the range of nice dimensions in the sense of Mather, then $\mu_\Delta(f) \geq \mathcal{A}_e - codim(f)$ with equality if f is weighted homogeneous.*

See Figure 1.2.

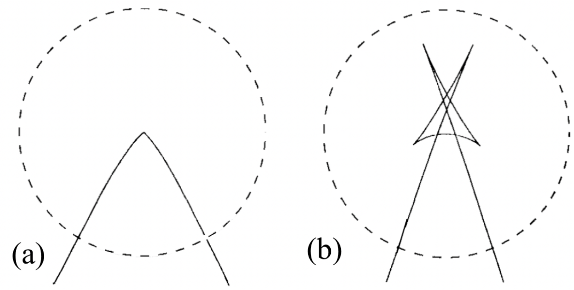


Fig. 1.2 (a) the real part of the discriminant of $f : (x, y) \mapsto (x, xy + y^6)$ and (b) the real part of the discriminant of a stable perturbation. We have $\mathcal{A}_e$ codimension of $f = \mu_\Delta(f) = 6$. In this case the real discriminant carries the vanishing homology of the complex discriminant.

This is in surprisingly close analogy to the much earlier results for functions and for ICIS, where the Milnor fibre also has the homotopy type of a wedge of spheres, and where the Milnor number μ (the number of spheres in the wedge), and the Tjurina number τ (the $\mathcal{K}_e$-codimension of the singularity), bear exactly the same relation as μ_Δ and $\mathcal{A}_e$-codimension in 1.1.9.

For a hypersurface singularity $(X, 0)$, K. Saito showed in [Sa71] that $\mu = \tau$ *only if* X is weighted homogeneous (with respect to some coordinate system), and the same result for an ICIS was shown by H. Vosegaard in [Vo]. It is not known whether the analogous result holds in the situation of Theorem 1.1.9. If it does, it will be only in the nice dimensions, where, indeed, every stable germ is weighted homogeneous with respect to a suitable coordinate system ([MN, 7.4]). It is striking that the proof of 1.1.9 makes explicit use of this fact. Outside the nice dimensions, not all stable germs are weighted homogeneous.

There is a simple explanation for the fact that the discriminant $D(f_t)$ of a stable perturbation f_t has the homotopy type of a wedge of $(p-1)$-spheres, even though discriminants are very far from having isolated singularities. The reason is that because f is $\mathcal{A}$-finite, it has an "isolated instability" at $0 \in \mathbb{C}^p$, so that when it is perturbed, the only change in its discriminant takes place in the immediate vicinity of the unstable point. From this, it follows that as f is deformed, the defining equation for $D(f_t)$ gains a certain number of isolated singular points which split off from $D(f_t)$. A Morse-theoretic argument of Dirk Siersma (see [MN, 8.3]) shows that each non-degenerate critical point contributes one sphere to the wedge, so that the number of spheres is the sum of the (ordinary) Milnor numbers of the isolated critical points.

In fact 1.1.9 is a special case of a much more general result of Jim's. There had been some other results on non-isolated singularities of functions, e.g.

by Siersma and Pellikaan, but it was Jim who identified an important broad but natural class of highly non-isolated singularities which shared crucial properties with isolated hypersurface singularities. This was his class of "almost free divisors", which included discriminants of finitely determined map-germs, bifurcation sets of certain unfoldings of hypersurface singularities, and many important hyperplane arrangements and non-linear hypersurface arrangements.

Jim arrived at this idea by considering non-linear sections of singularities, as in his papers 27 and 35. Any map-germ f with finite $\mathcal{K}$–codimension has a stable unfolding F, from which it can be pulled back by transverse fibre product, via a map g from the target of f to the target of F, as described in Subsection 1.1.2. Then the discriminant $D(f)$ of f is the preimage under g of the discriminant $D(F)$ of F. Since g need not be linear, $D(f)$ is a 'non-linear section' of $D(F)$.

It is important for the proof of 1.1.9 that when $n \geq p$, the discriminant of a stable mapping $D(F)$ is a 'free divisor' (the notion was introduced by Kyoji Saito in [Sa80]) – the module $\mathrm{Der}(-\log D(F))$ of smooth ambient vector fields tangent to $D(F)$ is locally free over the sheaf of germs of smooth functions. This allows the number of spheres in the wedge to be calculated easily, as the dimension of a certain quotient closely related to (1.3) below (this is the main theorem of Paper 41; alternatively see [MN, 8.7]). Indeed Arnol'd [ArW] and Zakalyukin [Z1, Z2], in their studies of the evolution of wave fronts, had already laid the foundation for the analysis of vector fields and functions on spaces containing discriminant varieties.

Generalising the above, an *almost free divisor* $(V', 0) \subset (\mathbb{C}^p, 0)$ is, by definition, the preimage of a free divisor $(V, 0) \subset (\mathbb{C}^P, 0)$ by a smooth map $g : (\mathbb{C}^p, 0) \to (\mathbb{C}^P, 0)$ which is "transverse" to V outside 0. Since free divisors are in general highly singular, the notion of transversality employed here needs to be explained. Jim contrasted two possible definitions: transverse to the canonical Whitney stratification, or transverse to the distribution $\mathrm{Der}(-\log V)$. While the former can be used to prove Whitney equisingularity, the latter fits better with the algebra of smooth mappings. Jim refers to it as "algebraic transversality". For an almost free divisor, the failure of algebraic transversality at 0 is in fact measured by the quotient

$$\frac{\theta(g)}{tg(\theta_p) + g^*\big(\mathrm{Der}(-\log V)\big)} \tag{1.3}$$

which we have already discussed in Example 1.1.7(1). Indeed, it can easily be shown, using nothing more than Nakayama's Lemma, that g is transverse to the distribution $\mathrm{Der}(-\log V)$, in the sense that

$$d_0 g(T_0\mathbb{C}^p) + \{\xi(0) : \xi \in \mathrm{Der}(-\log V)\} = T_0\mathbb{C}^P,$$

if and only if the quotient (1.3) is equal to zero. The (analytic) Nullstellensatz then implies that isolated failure of transversality is equivalent to the finite dimensionality of (1.3) as vector space over $\mathbb{C}$. As remarked in Example 1.1.7(1), the vector-space dimension (1.3) is equal to the $\mathcal{A}_e$ codimension of the germ f obtained from F by transverse pull-back via g. This equality played a crucial role in Theorem 1.1.9.

In the AMS Memoir, Paper 47, Jim defined the "singular Milnor fibre" of an almost free divisor as the preimage (inside a suitable small ball) V_t of V, under a perturbation g_t of g which is "algebraically transverse" to V. Again, this has the homotopy type of a wedge of spheres, yielding a "singular Milnor number". He extended Teissier's definition of higher multiplicities to this situation, and was able to calculate them using his Bézout Theorem for determinantal modules, for a range of important examples. It is typical of Jim's work that he found exactly the right context in which one can make these extensions of classical results, and, as with much of his work, it is probably not as well-known or appreciated as it might be.

Jim continued working on free and almost free divisors for many years. In Paper 53, he noted a remarkable phenomenon by which one obtains new free divisors from old. If $V_0 \subset (\mathbb{C}^p, 0)$ is an almost free divisor, the preimage of a free divisor $V \subset (\mathbb{C}^P, 0)$ by a map $g : (\mathbb{C}^p, 0) \to (\mathbb{C}^P, 0)$, then by the versality theorem for geometric subgroups 1.1.6, g has a $\mathcal{K}_{V,e}$ versal unfolding, $G : (\mathbb{C}^p \times \mathbb{C}^d, (0,0)) \to (\mathbb{C}^P \times \mathbb{C}^d, (0,0))$, which can be constructed by picking generators φ_i for the quotient (1.3) and defining $G(x,u) = (g(x) + \sum_i u_i\varphi_i, u)$. As usual, we write $G(y,u) = (g_u(y), u)$. In $\mathbb{C}^p \times \mathbb{C}^d$ we define the "relative critical set" Σ_G by

$$\Sigma_G = \{(y,u) : g_u \text{ is not transverse to } V \text{ at } y\},$$

and then define the "discriminant" of the unfolding, D_G, as the image of Σ under the projection $\pi : \mathbb{C}^p \times \mathbb{C}^d \to \mathbb{C}^d$. As in the case of smooth maps, Jim showed that Σ_G has dimension $d-1$ and, as π is finite on Σ_G, D_G is a divisor (a hypersurface) in $(\mathbb{C}^d, 0)$. The main theorem in Paper 53 is the following.

Theorem 1.1.10 *Under these circumstances, D_G is itself a free divisor, provided that the 0'th Fitting ideal of $\mathcal{F}_0(N\mathcal{K}_{D,un,e})$, as $\mathcal{O}_d$-module, is radical.*

This theorem has an immediate application in Thom-Mather theory: when $n \geq p$, the bifurcation set in the base of a versal unfolding of an $\mathcal{A}$-finite map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ can be viewed as one of the discriminants D_G just described. As explained below, in Paper 53 Jim determined the range of dimensions in which the hypothesis on the Fitting ideal held, and, thus, the range of dimensions in which the bifurcation set in the base of an $\mathcal{A}_e$ versal unfolding of a map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a free divisor. (The bifurcation set is the set of points u in the base-space of a versal unfolding $F(x,u) = (f_u(x), u)$ such that f_u is not stable.)

The module $N\mathcal{K}_{V,\mathrm{un},e}G$ needs some explanation, as does the condition on its 0'th Fitting ideal. $N\mathcal{K}_{V,\mathrm{un},e}G$ is the relative version of the quotient (1.3), in which g is replaced by its $\mathcal{K}_V$-versal unfolding G:

$$N\mathcal{K}_{V,\mathrm{un},e}G = \frac{\theta(G)}{tG(\theta_{p+d/d}) + G^*\mathrm{Der}\big(-\log(V\times\mathbb{C}^d)\big)} \tag{1.4}$$

where $\theta_{p+d/d}$ is the set of vector fields on $\mathbb{C}^{p+d}$ with zero component in the $\mathbb{C}^d$ direction. Since $\mathrm{Der}(-\log(V\times\mathbb{C}^d))$ contains the vector fields $\partial/\partial u_1,\dots,\partial/\partial u_d$ (where $u_1,\dots,u_d$ are coordinates on $\mathbb{C}^d$), if we write $G(x,u)$ as $(\bar{g}(x,u),u)$, we have

$$N\mathcal{K}_{V,\mathrm{un},e}G \simeq \frac{\theta(\bar{g})}{t\bar{g}(\theta_{p+d/d}) + \bar{g}^*\mathrm{Der}(-\log V)} \tag{1.5}$$

which evidently reduces to (1.3) when u is set to 0. The "N" in its name comes from "normal space to the orbit". This module is often referred to as the "relative T^1 of G" for $\mathcal{K}_V$-equivalence.

Since g is $\mathcal{K}_V$-finite, the projection from Σ_G to $\mathbb{C}^d$ is a finite map, and so $N\mathcal{K}_{V,\mathrm{un},e}G$ is also a finitely generated $\mathcal{O}_d$-module. Jim shows it has a presentation over $\mathcal{O}_d$ of the form

$$0 \longrightarrow \mathcal{L} \longrightarrow (\mathcal{O}^d)^d \stackrel{\alpha}{\longrightarrow} N\mathcal{K}_{V,\mathrm{un},e}G \longrightarrow 0 \tag{1.6}$$

where α send the i'th basis vector of $(\mathcal{O}_d)^d$ to $\partial\bar{g}/\partial u_i$, and $\mathcal{L}$ is the Lie algebra of vector fields on $\mathbb{C}^d$ which are liftable with respect to π to vector fields on Σ_G. Moreover, $\mathcal{L}$ is a free module, necessarily of rank d, because $N\mathcal{K}_{V,\mathrm{un},e}G$ is Cohen-Macaulay as an $\mathcal{O}_d$ module. Thus (1.6) can be replaced by

$$0 \longrightarrow (\mathcal{O}_d)^d \stackrel{\lambda}{\longrightarrow} (\mathcal{O}^d)^d \stackrel{\alpha}{\longrightarrow} N\mathcal{K}_{V,\mathrm{un},e}G \longrightarrow 0 \tag{1.7}$$

where the columns of the matrix λ are the components of the d vector fields generating $\mathcal{L}$.

For any ring R and R-module with a finite presentation, the *zero'th Fitting ideal of M* is the ideal generated by the maximal minors of the matrix of a presentation. Thus, here it is the ideal generated by $\det\lambda$. It is geometrically evident that liftable vector fields are tangent to D_G, i.e. that $\mathcal{L}\subseteq\mathrm{Der}(-\log D_G)$; if the two are equal, and if $\det\lambda$ is reduced, then D_G is a free divisor, by K. Saito's famous criterion ([Sa80]). In paper 53, Jim gave sufficient conditions for this to hold.

The algebraic condition has a geometrical meaning, that π should map Σ_G isomorphically to D_G "at most points". In effect, this is asking that at "generic" points where g_u fails to be transverse to D, the quotient (1.3) should have vector space dimension 1. Jim referred to these conditions for D_G to be a free divisor by saying that non-linear sections of D should generically have "Morse-type" singularities (Definition 4.1 in Paper 53). Applying his result to the bifurcation set in the base of an $\mathcal{A}_e$-versal unfoldings of a map-germ $(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ as explained above, in Section 6 of Paper 53 he determined the range of dimensions (n, p) for which the condition on Morse-type singularities holds.

Jim's theorem 1.1.10 reveals an intriguing vista of sequences of free divisors, each obtained from its predecessor by the procedure he describes. The structure of these sequences, including possible periodicities, remains to be understood.

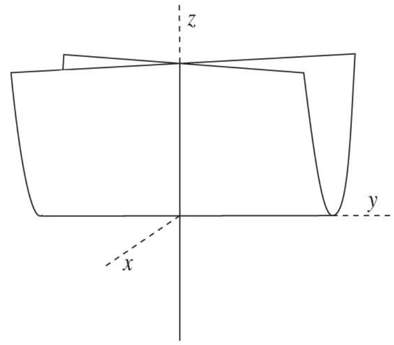
With his student Brian Pike, in Papers 82, 86 and 87, he worked towards describing the Milnor fibres of determinantal singularities, building on the methods he developed on almost free divisors. This work makes use of the unexpected abundance of free divisors arising from group actions ("linear free divisors") recently revealed by work of Buchweitz and others ([BuMo], [GMNS]).

His last papers on non-linear sections of singularities, Papers 94, 97 and 100 (in press) broadened the focus to discuss the topology of the Milnor fibre F_{V_0} – the nearby non-singular level set – of a non-linear section $V_0 = g^{-1}(V)$ of a singular hypersurface V. This is distinct from the "singular Milnor fibre" he had introduced in Paper 47. In Paper 97 Jim introduced and studied the "characteristic cohomology" of the Milnor fibre of V_0, namely the subalgebra $g_t^*(H^*(F_V))$ of the cohomology pulled back from the cohomology of the Milnor fibre F_V of V, and likewise the characteristic local cohomology of the complement. These ideas suggest a route towards a description of the cohomology of the Milnor fibres of many of the non-isolated singularities which occur naturally in geometry, such as the determinants of families of matrices. In Paper 100 he applied it to the "matrix singularities" studied by Bruce, Goryunov and others (as described in Example 1.1.7(4) above).The Kato-Matsumoto theorem, that unlike the Milnor fibre of an isolated singularity, the Milnor fibre of a non-isolated singularity will in general have homology in a range of dimensions below middle dimension, shows the formidable difficulties involved, and Jim's ideas may well bear fruit in this task. Like much of his work, it opens doors for other mathematicians; much work remains to be done.

1.1.4 Semi-coherence

We've seen that the basic theorems of singularities of map-germs (including the finite determinacy theorem, versal unfolding theorem, and classification theorems) work for any "geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$". For example, in the real or complex analytic category, the group of bianalytic-germs preserving an analytic set is geometric in the above sense; this extends to the smooth case for a set diffeomorphic to a real coherent analytic set, as defined by Malgrange: the analytic set V is *coherent* at 0 if any finite set of generators of the ideal $I(V_0)$ of analytic germs at 0 vanishing on V has extensions generating $I(V_x)$ for all x in V in a sufficiently small neighborhood of 0.

Example 1.1.11 The Whitney umbrella $x^2 - y^2 z = 0$ is not coherent, as $x^2 - y^2 z$ does not generate $I(V_p)$ for $p(0,0,z)$ $z < 0$, on the handle of the umbrella, the negative z-axis.



A subset of $\mathbb{R}^n$ is *semialgebraic* if it is a finite union of sets defined by polynomial equations and inequalities, and *semianalytic* if, for each $x \in \mathbb{R}^n$, there exists a neighborhood U of x, such that $A \cap U$ is a finite union of sets defined by real analytic equations and inequalities. The image of the map $(u, v) \mapsto (uv, u, v^2)$ is semialgebraic: it is the part of the Whitney umbrella with $z \geq 0$. In Paper 88 Jim defined and studied notions of semi-coherent (see Definition 1.1.12 below) semianalytic sets and semi-coherent semianalytic stratifications, and proved that the group of diffeomorphisms preserving such a set is "geometric". This is a broad generalization of the previously existing results. Several examples where these conditions play a role involve: discriminants of stable germs, which in general are only (diffeomorphic to) semialgebraic sets; the Blum medial axis (or central set) for generic smooth regions in $\mathbb{R}^n$ are locally diffeomorphic to semialgebraic sets, and in computer vision, the stratifications which are needed to describe the geometric features of natural objects, and the refinements of these stratifications resulting from shade and shadows, requires the consideration of semianalytic stratifications. See §1.2 and §1.3 below.

Definition 1.1.12 A closed semianalytic set germ $V, 0 \subset \mathbb{R}^n, 0$ (with Zariski closure $\overline{V}$) with a given semianalytic stratification will be said to be *semi-coherent* in the smooth category if the following two conditions are satisfied:

i) The smooth vanishing ideal of V equals $\mathcal{E}_n\{g_1, ...g_s\}$ mod m_n^∞, where $\{g_1, ...g_s\}$ generate $I(\overline{V})$ ($\mathcal{E}_n$ = smooth germs and m_n^∞ denotes the ideal of infinitely flat smooth germs), and

ii) An analogous condition holds for germs of smooth and analytic vector fields tangent to the stratification of V.

Every semianalytic set has a canonical semianalytic Whitney stratification, and V is said to be semi-coherent if the above definition holds for that stratification.

More generally, V is semi-coherent if it is smoothly diffeomorphic to a semi-coherent semianalytic set.

Example 1.1.13 The need for the infinitely flat terms in the definition above is illustrated by the semianalytic set V given by the union of the positive x- and y- axes in $\mathbb{R}^2$. Let $f(x) = e^{-1/x^2}$, $x < 0$, and $f(x) = 0$, $x \geq 0$, and $g(y)$ defined similarly. Then $F(x, y) = f^2(x) + g^2(y)$ is zero on the closed first quadrant, including V, and is positive elsewhere. Since $F(x, y)$ is positive on the negative x- and y- axes, it can't be in the smooth ideal generated by xy, which generates $I(\overline{V})$.

Jim showed that a broad class of semianalytic sets are semi-coherent, including:

a) any semianalytic set V whose Zariski closure is weighted homogeneous for positive weights with V invariant under the corresponding $\mathbb{R}_+$ action;

b) the discriminant set of a simple smoothly stable germ.

From a) we see that the Whitney Umbrella is a non-coherent analytic set which is semi-coherent.

1.1.5 Application of singularity theory to bifurcation theory

Bifurcation theory generally refers to the study of changes of equilibrium solutions of functional or of ordinary or partial differential equations depending on parameters. The parameters may vary over an infinite dimensional space. Various techniques are used (Liapunov-Schmidt, Center Manifold, Implicit Function Theorem) to reduce the study to a finite dimensional local model $f : \mathbb{R}^{n+q}, 0 \to \mathbb{R}^p, 0$. This is viewed as a perturbation $f_u(x) = f(x, u)$, using parameters $u \in \mathbb{R}^q$, of a germ $f_0(x) = f(x, 0)$ of a smooth mapping at 0, $f_0 : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$.

There is a variety of notions of equivalence for studying such perturbations. In Paper 31, Jim considers:
1) *simple bifurcation*, which studies the variation of $f_u^{-1}(0)$;
2) *imperfect bifurcation*, introduced by Golubitsky and Schaefer [GS] to study bifurcation in the presence of a distinguished parameter;
and also the following which we won't describe: *sequential bifurcation*, *multi-parameter bifurcation*, and *equivariant bifurcation*.

For each of these, a notion of equivalence is introduced for germs of mappings and their unfoldings. For simple bifurcation, the group is contact equivalence (i.e. $\mathcal{K}$). For imperfect bifurcation equivalence, we choose local coordinates x, λ for $\mathbb{R}^{n-1}, \mathbb{R}^1$ and use the subgroup $\mathcal{K}'$ of $\mathcal{K}$ equivalences with $h(x, \lambda) = (h_1(x, \lambda), h_2(\lambda))$. This group still acts on $\mathcal{E}_n^p$, although p is usually taken to be $n - 1$. For this and the other cases it is proven that the groups are geometric subgroups of $\mathcal{K}$, as in §1.1.2, so the theorems of that section can be applied to solve the equivalence problem, the determinacy problem and the versality problem. Frequently however one runs into moduli which parametrize continuous change in the smooth equivalence classes of such problems. To overcome this difficulty it is necessary to replace each notion of equivalence by its corresponding topological analogue and to answer the above questions for the topological versions of equivalence – see Jim's paper 30.

Most of Paper 30 is concerned with the computation of triviality and versality of unfoldings using weightings of the variables. Let $\mathcal{K}^*$ be any of the bifurcation groups on $\mathcal{E}_n^p$, fix weights $a_1, \ldots, a_n$ on $\mathbb{R}^n$ and $d_1, \ldots, d_p$ on $\mathbb{R}^p$, let P denote the set of weighted homogeneous polynomial germs in $\mathcal{E}_n^p$ of the given weights and of finite $\mathcal{K}^*$-codimension . Then

Theorem 1.1.14 *Let g_0 and f_0 in $\mathcal{E}_n^p$ be germs whose initial parts (terms of lowest weight) both lie in the same path component of P. Then g_0 and f_0 are topologically $\mathcal{K}^*$-equivalent.*

Example 1.1.15 Let $f_a(x, \lambda) = x^3 - 3ax^3\lambda + \lambda^3$ (from the classification of [K] of one-variable imperfect bifurcation problems of codimension ≤ 7). This has finite $\mathcal{K}'$-codimension if $a \neq (1/4)^{1/3}$. It is weighted homogeneous with $wt(x, \lambda) = (1, 1)$. By the above theorem there are (at most) two topological equivalence classes corresponding to $a < (1/4)^{1/3}$ and $a > (1/4)^{1/3}$.

Paper 30 gives many other examples.

1.1.6 Generic properties of singularity theory to solutions of partial differential equations

For most partial differential equations (often with boundary conditions), it is not feasible to find all solutions, and one may instead seek to understand qualitative properties of generic solution. In Paper 49 Jim tackles this problem using the methods of singularity theory. In particular he seeks answers to the questions:

(1) What are generic properties of solutions of a given equation, and are they exhibited for a generic set of boundary conditions?

(2) How can generic properties be established for particular types of equations?

(3) How can geometric properties of solutions be deduced?

Unfortunately the constraint of the equation can render standard genericity and transversality arguments invalid, so that new versions of these results are needed that will apply to the PDE context. Jim proves two extensions of the Thom transversality theorem. The "Relative Transversality Theorem" allows one to find solutions of a PDE problem which are generic (with respect to some property being considered) amongst all solutions of the problem. The "Absolute Transversality Theorem" then gives solutions which are generic amongst all functions if the family of all solutions has certain properties. These theorems allow Jim to characterize geometrically the generic properties of solutions, as well as to establish the stability of these properties. Two general classes of partial differential operators are introduced: (i) weighted homogeneous and (ii) filtered.

A linear operator $P : \mathcal{E}_n \to \mathcal{E}_p$ is called *weighted homogeneous of weight* r if P maps functions of weight l to functions of weight $l + r$. If a variable x has weight l then the differential operator $\dfrac{\partial}{\partial x}$ has weight $-l$. The Laplace operator Δ and the wave operator $D_t^2 - c^2\Delta$ are weighted homogeneous of weighted degree -2 with $wt(x_i) = wt(t) = 1$, and the heat operator $D_t - k\Delta$ is weighted homogeneous of weighted degree -2 with $wt(x_i) = 1$ and $wt(t) = 2$. Some non-constant coefficient examples are: the Lewy operator $D_{\overline{z}} - izD_t$ is weighted homogeneous of weighted degree -1 with $wt(z, \overline{z}, t) = (1, 1, 2)$ and the Euler-Tricomi operator $D_x^2 - xD_y^2$ is weighted homogeneous of weighted degree -4 with $wt(x, y) = (2, 3)$. For weighted homogeneous operators, the local structure of solutions is given in terms of normal forms involving weighted homogeneous solutions. Global genericity holds for solutions to weighted homogeneous equations provided they have at least one smooth solution

Most operators are not weighted homogeneous, but (nonlinear) filtered differential operators are characterized by their behavior relative to weight filtrations. A local approximation theorem is proved which allows one to deduce the local existence of solutions with given lower-order Taylor expansions. Sufficient conditions for differential operators are given so that the transversality theorems are applicable to the the associated solution spaces. The conditions for the spaces of local solutions and the space of global solutions are related by a "Stein condition" analogous to that characterizing Stein manifolds in terms of jets of local functions being realized by global functions.

The methods are illustrated by several applications, including caustics, shocks and problems in computer vision where an initial pixel intensity function u_0 is "Gaussian blurred" via a nonlinear heat equation. Features of u_0 are characterized by geometric properties of the resulting solution u. This was the focus of Jim's earlier work (Paper 43), now seen as a special case of the more general theory presented here.

1.2 Medial structures: Peter Giblin

Jim Damon's work in the area of medial structures made a profound impact not only on the mathematical theory but on its applicability to "real world" situations, in particular to medical image computing, as described in Stephen Pizer's contribution to this chapter (§1.4). In this section I shall describe the mathematical basis for these developments, starting with a brief introduction to the "classical" Blum medial axis and proceding to Jim's more flexible definition and subsequent extensions and generalisations to skeletal structures. All of these developments, which include a good deal of profound mathematics, led to a greater understanding of the underlying goal: how can we encode the shape of an object by replacing it with some kind of "skeleton" of lower dimension?

1.2.1 The Blum medial axis: examples

The origins of these ideas are in [BN, B].

Definition 1.2.1 The Blum medial axis of a smooth closed plane curve C, or of the region bounded by C, is the closure of the locus of centres of disks contained in this region and whose bounding circles are tangent to C in at least two places—"bitangent circles".

It is usual to talk slightly loosely by repacing "disks" with "circles", understanding that the whole circle, including its interior, must lie within the region bounded by C, and the *circle* is tangent in at least two places to C.

The definition for surfaces is exactly parallel:

Definition 1.2.2 The Blum medial axis of a smooth closed surface S in $\mathbb{R}^3$, or of the region bounded by S, is the closure of the locus of centres of 3-disks (3-balls) contained in this region and whose bounding 2-spheres are tangent to S in at least two places—"bitangent spheres".

Again, it is usual to talk of centres of spheres contained in the region, meaning the whole sphere and its interior are contained therein. In both definitions, 'closure' is needed to allow for limiting cases where contact points with the boundary C or S coincide.

Remark 1.2.3 The *symmetry set* of C or S above is obtained by dropping the requirement that the disk or 3-disk is contained within the region bounded by C or S. Its interest is more theoretical than the medial axis; for further information see for example [GK, Gib] and for earlier mathematical work [Yom, BGG]. The main mathematical tool used in the investigation is the *full bifurcation set* of a family of functions, namely the family of distance-squared functions on C or S. This approach is different from the one adopted in the present chapter, which follows Jim's publications and leads to much more general results.

The contact functions between curve/surface and circle/sphere are labelled by the standard "A_k" notation—note that only *odd* k can arise since for even k the circle/sphere will *cross* the curve/surface and therefore cannot be wholly inside it. These are illustrated in the Figures 1.3, 1.4, 1.5 and 1.6. The medial axis is denoted by M below. The following are the generic cases arising.

(i) A_1^k: ordinary tangency at k distinct points, $k = 2, 3$ for the 2d case and $k = 2, 3, 4$ for the 3d case. In both cases, $k = 2$ corresponds to a smooth point of the medial axis. In the 2d case, $k = 3$ gives a "triple point" on M. In the 3d case $k = 3$ gives a "Y-junction (or Y-branch) curve" where three sheets of the medial axis intersect, and $k = 4$ a "6-junction point" where four sheets meet in $\binom{4}{2} = 6$ curves (or four Y-junction curves).

(ii) A_3: In the 2d case this means the centre of the circle is also the centre of curvature of C at the single contact point, which is a vertex of C and an end point of M. In the 3d case it means that the centre of the sphere is one of the centres of principal curvature of S at the contact point which is an edge point of M, and a "ridge point" or "crest point" of S (see for example [BGT]).

(iii) A_1A_3: In the 3d case the centre of the sphere is a centre of principal curvature at one contact point and the sphere has ordinary contact at another point. This is a "fin creation point" on M where a Y-junction curve meets an edge curve.

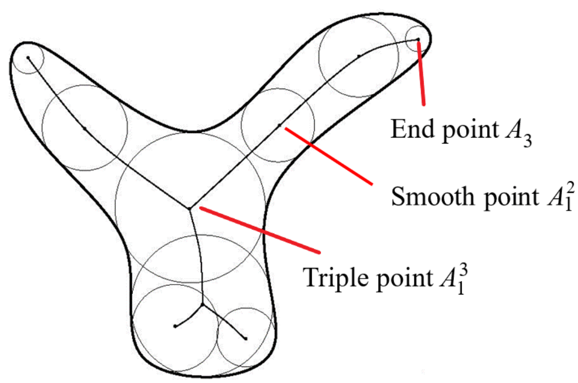


Fig. 1.3 The 2d case, showing a triple point and an end point on the medial axis. These, together with the smooth (A_1^2) case, are the only generic cases for a boundary which is a plane curve.

1.2.2 Radial vector fields and shape operators in the Blum case

Notation From now I shall confine the discussion to the 3d case and bitangent 2-spheres, and adopt Jim's notation $\mathcal{B}$ for the boundary surface S.

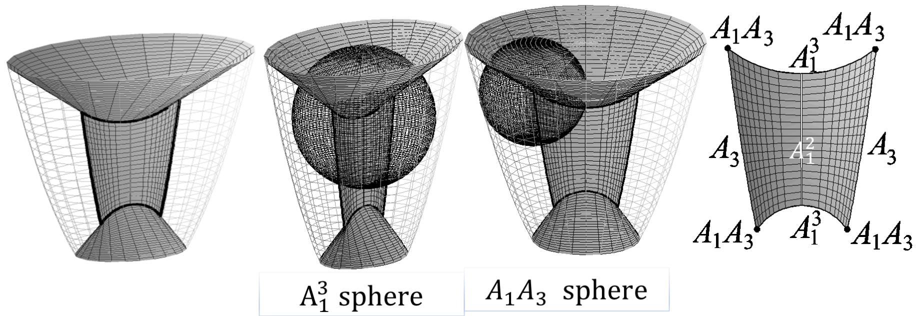


Fig. 1.4 Left to right: an "elliptical cylinder" (with expanding ellipses bottom to top), having a lid and a base and shown in pale wire-frame. (This is strictly not a smooth surface but sufficent for this illustration.) The medial axis is darker. Next, a sphere with three ordinary contacts (top and two points on the curved surface); a sphere with one ordinary contact (at the top) and centre at a fin creation point A_1A_3; and (far right) the various contacts of spheres centred on the medial axis. For the A_1^4 case see Figure 1.5. The whole interior of the medial axis sheet consists of A_1^2 points, corresponding to ordinary contact of a sphere at two distinct points.

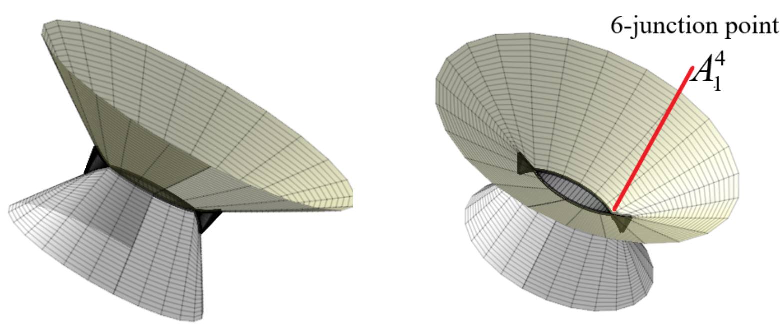


Fig. 1.5 Medial axis of a shorter elliptical cylinder with base and lid (the latter not shown) in two semi-transparent views with the medial axis rendered semi-transparent. Now there exist two spheres tangent in four places (top, bottom, front and back) whose centres are at the A_1^4 crossing points of the dark curve. The A_1^2 region in the rightmost diagram of Figure 1.4 has contracted so that the two A_1^3 curves intersect. This has resulted in a "horizontal" 2-sided region flanked by two "vertical"' cusped 3-sided regions, seen best in the right-hand view above.

Jim started from the basic ideas above and transformed them into a powerful and original general mathematical theory which had many applications to, for example, medical image computing and shape analysis. I shall sketch how this was done and quote some of the results obtained along the way. After that I shall explain the extension to more general "skeletal structures"

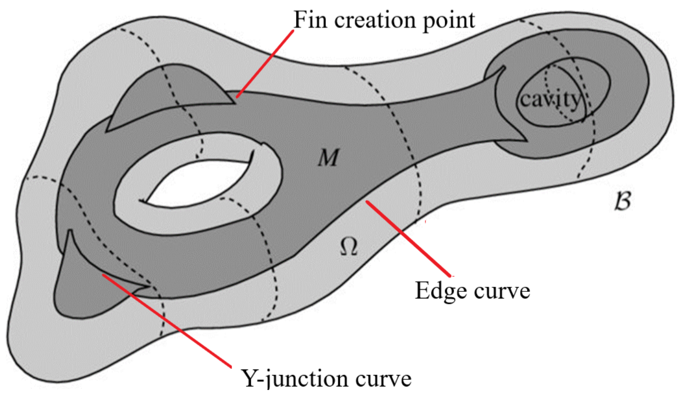


Fig. 1.6 Medial axis (dark) showing Y-junction curves (A_1^3), fin creation points (A_1A_3) and edge curves (A_3). The boundary surface S is here and below denoted by $\mathcal{B}$. Figure adapted from Figure 4 of Paper 68.

and the correspondingly more general results relating the geometry of the boundary $\mathcal{B}$ to properies of these structures. The main reference for this section is Paper 65, though the ideas are also contained in the longer expository article Paper 77 and proved in the two articles, Papers 63 and 64. In Paper 77 the notation $\mathbf{V}$ is used for U below and $\mathbf{U}$ for U_1 below.

The part of the Blum medial axis M which is smooth (A_1^2 points) is denoted M_{reg} and the non-smooth parts are, as above, the Y-junction curves, edge curves, fin creation points and 6-junction points. These are denoted M_{sing}. In fact M as a whole is a Whitney stratified set. On M there is defined a multivalued *radial vector field* U from points $P \in M$ to all the contact points of the sphere centred at P with $\mathcal{B}$; the *radial distance function* r is then $||U||$. We denote the corresponding unit vector field by $U_1 = U/r$. At points of M_{reg}, U has two values and at points of M_{sing} it has one value at edge (A_3) points, two values at fin creation (A_1A_3) points, three at Y-junction (A_1^3) points and four at 6-junction (A_1^4) points.

For the Blum medial axis note that the radial vector U is orthogonal to the boundary surface $\mathcal{B}$. This is called the *partial Blum condition* for the pair (M, U). (There are other special properties of U in the Blum case, such as the two vectors U at points of M_{reg} being of the same length and making equal angles with the tangent plane to M.)

Among the "classical" interpretations ([B, KTZ]), of the medial axis is the "grassfire flow", that is flow at unit speed along the normals to $\mathcal{B}$ starting from $\mathcal{B}$ itself and moving into the the region bounded by $\mathcal{B}$. The resulting surfaces are then parallels to the boundary and initially smooth but become singular when the medial axis is reached, at which the parallels have self-intersections. Jim used the opposite flow, called the *radial flow*

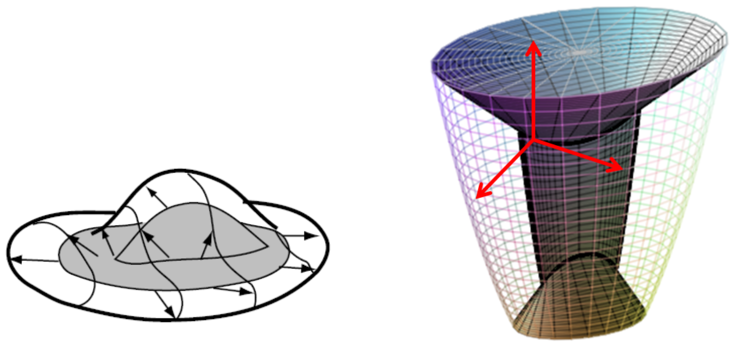


Fig. 1.7 Left: Radial vector field on a medial axis (dark). This is adapted from Figure 1 in Paper 63. Note that from smooth parts of the medial axis there will be two radial vectors, one occluded and pointing "downwards"; from edge point just one radial vector as shown; from Y-junction points three, only one being shown; and from fin points, where edges meet Y-junction curves, two, one being shown. On the right: 3 arrows from a Y-junction point for the surface in Figure 1.4 above. Of course all the arrows are of equal length, being radii of a fixed sphere.

$$\psi_t(x) = x + tU(x) \tag{1.8}$$

where $x \in M$, which at time $t = 1$ reaches the boundary $\mathcal{B}$, the point of $\mathcal{B}$ depending on the choice of $U(x)$. See §1.1 of Paper 65. Note that the level sets of this flow are not smooth until $t = 1$ and the speed of the flow depends on the radius function r.

A crucial tool introduced by Jim Damon is the *Radial Shape Operator*, defined in a way analogous to the Shape Operator of the differential geometry of surfaces (see for example [O'N, Ch.5]). This operator is the key to relating the geometry of M to that of the boundary surface $\mathcal{B}$. I need to emphasize that Jim makes these definitions in n dimensions, but I shall keep to the case of surfaces to simplify the exposition.

Definition 1.2.4 The radial shape operator $S_{rad}(v)$ at $x_0 \in M$ assigns to each tangent vector v to M at x_0 the projection into the tangent plane at x_0, along U, of the vector $-\frac{\partial U_1}{\partial v}$. This latter vector (without the minus sign) can also be written $\nabla_v U_1$ (compare [O'N, Ch.5]), the directional derivative of the unit vector field U_1 in the direction v.

Thus using local coordinates $X(u_1, u_2)$ on M near x_0, so that $v_i = \frac{\partial X}{\partial u_i}, i = 1, 2$ evaluated close to x_0 span the tangent plane to M at points close to x_0, we write

$$\frac{\partial U_1}{\partial u_i} = a_i U_1 - b_{1i} v_1 - b_{2i} v_2$$

and the matrix of $S_{rad}(v)$ is

$$S_v = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Notation The notation $S_{\mathbf{v}}$ will be used to denote the shape operator with respect to the basis $\mathbf{v} = \{v_1, v_2\}$.

Definition 1.2.5 The *principal radial curvatures* at x_0 are the eigenvalues κ_{r1}, κ_{r2} of S_v, and the *principal radial directions* are the corresponding eigenvectors. These are independent of the choice of basis for the tangent plane.

The above definition extends to non-smooth points of M except edge points, by continuing the sheet of M smoothly beyond a Y-junction point for example (this extension being in fact continuing on a sheet of the symmetry set).

Edge points (A_3 points) present a problem in the smoothness of the vector field U. For a very simple example of this consider the "parabolic trough" $z = y^2$ illustrated in Figure 1.8, left. Two contact points of a bitangent sphere are shown; as the sphere shrinks and moves downward these points come into coincidence and the centre of the sphere becomes a principal centre of curvature of the surface at the single contact point. The medial axis consists of the points $(a, 0, \frac{1}{2}+t)$ for $t \geq 0, t = 0$ giving the edge, in a similar way to the left and right edges of Figure 1.4. The radial vector joining this point to the corresponding points on the boundary surface $z = y^2$ is $U = (0, \pm\sqrt{t}, t)$ and because of the square root sign this is not a smooth vector field at $t = 0$. The cure for this problem is to split U into two parts, corresponding to the two "sides" of the medial axis, parametrizing these by $(0, u, u^2)$ and $(0, -u, u^2)$ which are related by a local diffeomorphism near $u = 0$ so that altogether U is smooth, and hence so is the radius function $r = ||U||$. This is called taking *edge coordinates* and generalises to any edge curve. It allows us to differentiate the unit vector field U_1 as in the definition below.

For another example and further explanation see Example 1.4 in Paper 63.

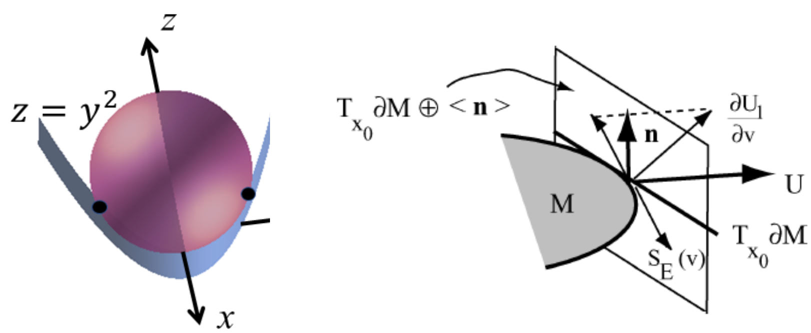


Fig. 1.8 Left: a parabolic trough (see text). Right: the edge shape operator construction, where the dashed line represents projection onto the plane of the tangent to ∂M and the normal vector $\mathbf{n}$ to M (from Figure 9 in Paper 65).

The revised definition of the shape operator required for edge points is as follows (Figure 1.8, right).

Definition 1.2.6 Let x_0 be an edge point of M and, as usual, U the unique vector from x_0 to the contact point with $\mathcal{B}$ of corresponding sphere. Let $\mathbf{n}$ be a unit normal vector field to M and v a vector in $T_{x_0}\partial M \oplus \mathbf{n}$; then, using edge coordinates to perform the differentiation, the edge shape operator is defined by $S_E(v) = -\text{proj}' \frac{\partial U_1}{\partial v}$ where proj$'$ means projection onto $T_{x_0}\partial M \oplus \mathbf{n}$ along U. (Again $\frac{\partial U_1}{\partial v}$ can be written $\nabla_v U_1$.)

As before the edge shape operator can be given a matrix representation (Example 2.4 in Paper 65) and the single principal edge curvature at an edge point of M is then defined as a generalized eigenvalue of the pair consisting of this matrix and the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The principal edge curvature can also be obtained as a limit, avoiding the use of edge coordinates—see Proposition 3.7 in Paper 65 and Corollary 3.9 in Paper 64.

Of course, starting from M, U and r there is no guarantee that a constructed boundary $\mathcal{B} = \{x + rU\}$ will be smooth. Damon is able to show (Theorem 2.3 in Paper 63; Theorem 2.3 in Paper 65) that three conditions guarantee at any rate that $\mathcal{B}$ is smooth corresponding to $x \in M_{reg}$ and has well defined limiting tangent planes for $x \in M_{sing}$, with additional conditions implying $\mathcal{B}$ is an embedded surface. The conditions state that $r < \min \frac{1}{\kappa}$ for all principal positive radial and edge curvatures κ and that the "compatibility 1-form" $\eta_U(v) = U_1 \cdot v + dr(v)$ is identically 0 for all singular and edge points of M. I shall not expand here on these conditions; see the references for more details.

1.2.3 Differential geometry of the boundary $\mathcal{B}$

One of the objectives of Jim's work on medial axes (by no means the only objective) is to deduce the differential geometry of the boundary $\mathcal{B}$ from medial data, and the results here are set out in §3 of Paper 65, §3 of Paper 77 and Theorem 3.2 of Paper 64. Here is a sample of what can be proved. (See above for the notation.)

Theorem 1.2.7 *Let $x'_0 \in \mathcal{B}$ correspond to $x_0 \in M$ by the vector U and let $\mathbf{v}' = \{v'_1, v'_2\}$ be the image under $d\psi$ (see equation (1.8)) of a basis $\mathbf{v} = \{v_1, v_2\}$ of the tangent plane $T_{x_0}M$ where x_0 is a non-edge point of M (so that U extendes smoothly across a Y-junction curve for example, when $x_0 \in M_{sing}$). Then*
(i) The usual differential geometric shape operator $S'_{\mathcal{B}}$ (see for example [O'N, Ch.5]) at x'_0 has matrix representation with respect to $\mathbf{v}'$ given by

$$S'_{\mathcal{B}\mathbf{v}'} = (I - rS_{\mathbf{v}})^{-1} S_{\mathbf{v}}.$$

(ii) The principal radial curvatures κ_{ri} of M at x_0 and the principal curvatures κ_i of $\mathcal{B}$ at x'_0 are related by

$$\kappa_i = \frac{\kappa_{ri}}{1 - r\kappa_{ri}} \text{ that is } \kappa_{ri} = \frac{\kappa_i}{1 - r\kappa_i}.$$

Note that writing $r_i := 1/\kappa_i$ and $r_{ri} := 1/\kappa_{ri}$ for the signed radii of curvature *these state, more simply, that*

$$r_{ri} = r + r_i.$$

This can be regarded as a large generalization of the formula for the radii of curvature of two parallel surfaces separated by an orthogonal distance r.
(iii) The principal radial directions corresponding to κ_{ri} are mapped by the radial flow $d\psi_i$ to the principal directions corresponding to the κ_i.

1.2.4 A note on ridges on $\mathcal{B}$.

These are not to be confused with *height ridges* as occurring in §1.2.5 below and in §1.4; see §1.4.2.

A ridge (crest) point on a smooth surface $\mathcal{B}$ is a point where where there is a sphere with A_3 contact, corresponding to an edge point on M. These points lie on ridge curves such as the bottom of the trough in Figure 1.8, left, and the curved sides of the "cylinder" in Figure 1.4. There is an extensive discussion of ridge curves on surfaces in [Por, Patt] and conditions for the different kinds of ridges are given for example in the book [Patt, p.139]. An *elliptic ridge point* on a surface $\mathcal{B}$ is a point where one of the spheres of curvature has A_3

contact and meets the $\mathcal{B}$ in an isolated point; for a *hyperbolic ridge point* the contact is still A_3 but the intersection is two tangential curves (a "tacnode", Figure 1.9, left). Clearly for the sphere to contribute to the medial axis it must be locally (indeed globally) on one side of $\mathcal{B}$ so only elliptical ridge points can contribute to the medial axis. Each point of $\mathcal{B}$ has two spheres of curvature (coincident at umbilics) and it is possible for these to be distinct and both to meet $\mathcal{B}$ in an isolated point, but in that case they are locally on opposite sides of $\mathcal{B}$ so cannot both contribute to the medial axis. See Figure 1.9, right. (At a hyperbolic point of $\mathcal{B}$ the centres themselves are on opposite sides of the surface.)

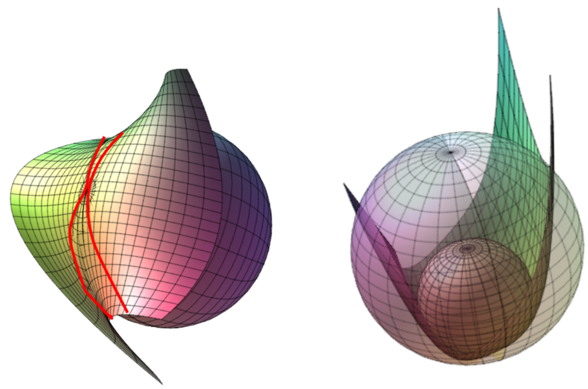


Fig. 1.9 Left: a sphere of curvature at a hyperbolic ridge point, meeting the surface in two tangential curves; right: two spheres of curvature both meeting the surface in an isolated point but one (the semi-transparent one) is locally "outside" the surface and the other is locally "inside" it.

1.2.5 Height ridges and other special curves

A different kind of ridge (a height ridge) was introduced by Pizer and Eberly and investigated, together with related curves, by Jim; see Papers 65 and 77 and §1.4.2 below. For this, consider any smooth surface written as a graph $z = f(x, y)$ and let $\lambda_1 < \lambda_2$ be the eigenvalues of the Hessian $H(f)$ at $p_0 = (x_0, y_0)$ with eigenvectors e_1, e_2. Then the point p_0 is called a

Definition 1.2.8 (i) height ridge point if the gradient $\nabla f(p_0)$ is orthogonal to e_1 and $\lambda_1 < 0$,
(ii) valley point if $\nabla f(p_0)$ is orthogonal to e_2 and $\lambda_2 > 0$,
(iii) r-connector point if $\nabla f(p_0)$ is orthogonal to e_1 and $\lambda_1 > 0$, and
(iv) v-connector point if $\nabla f(p_0)$ is orthogonal to e_2 and $\lambda_2 < 0$.

The *relative critical set* $\mathcal{RC}(f)$ is the closure of the above four sets together with critical points of f, points where one eigenvalue is 0, and points where

$\lambda_1 = \lambda_2$. The idea is that $\mathcal{RC}(f)$ carries much information about the "relative geometry" of the surface which is the graph of f. Figure 1.10 shows Damon's illustration of some of these curves.

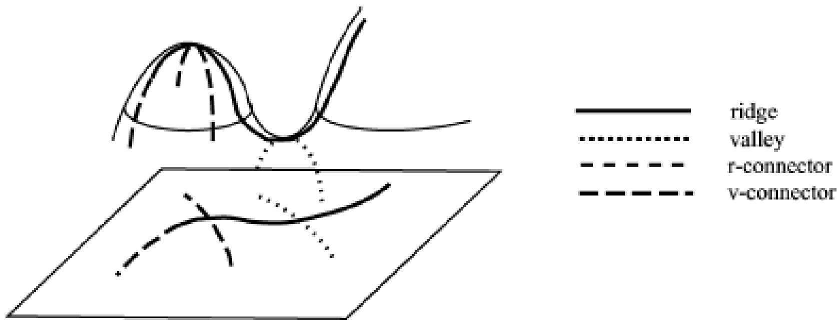


Fig. 1.10 (From Figure 12 in Paper 65). Ridges,valleys and connectors.

The main application of the above construction is to the function r on the medial axis, building on work of Eberly [E], so here it is the "relative geometry" of the boundary $\mathcal{B}$ and the medial axis M which is being studied. Among other things, Jim studied the effect of medial axis diffeomorphisms on the relative geometry; see Papers 65 and 77 for summaries of this work, and Paper 52 for stability properties. The general idea is that medial axes may have similar general flat-ish shape but the boundaries can vary considerably, rather like potatoes. An illustration from Figure 16 in Paper 77 is given in Figure 1.11.

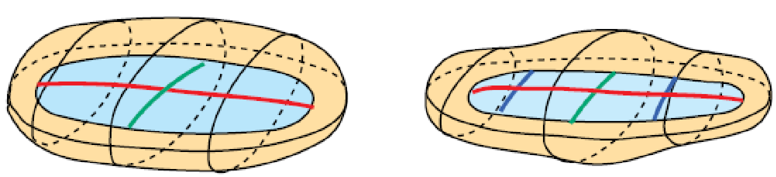


Fig. 1.11 Different geometry of boundaries distinguished by the networks of ridge, valley and connector curves.

1.2.6 Skeletal structures

In this section I shall briefly describe the generalization of the idea of a medial structure beyond the Blum case, that is dropping some of the constraints on the radius function r and radial vector field U, and then equally briefly point towards Jim's more recent work on multi-object configurations. See also §1.4.4 below.

A difficulty with medial axes is that starting with a smooth boundary and its associated medial axis, distorting the medial axis, the radius function or the vector field U may imply that any reconstructed boundary is no longer smooth. Note that this is distinct from the sensitivity of medial axes to distortions of the boundary—a small depression in the boundary creates a whole new structure on the medial axis—since it implies that medial axes themselves cannot be safely manipulated. Clearly Blum medial structures M are special in many ways: for example the various radial vectors emanating from a single point of M are all the same length, they are all perpendicular to the boundary (they are radii of bitangent spheres) and they make the same angle with the tangent plane to the medial axis. The concept of skeletal structure seeks to generalize beyond these constraints. I shall concentrate on the 2-dimensional case, analogous to the Blum medial axis of surfaces in $\mathbb{R}^3$, but as before Jim's results are dimension-independent.

The full technical details of the definition of skeletal structure are given in Paper 63, but the general idea is to start with a Whitney stratified set M, a union of disjoint smooth strata satisfying the usual Whitney conditions, with M_{reg} the union of the 2-dimensional strata, M_{sing} the union of the remaining strata, and ∂M the union of those points of M_{sing} at which M is locally a manifold with boundary (*edge* points). Fin creation points in the Blum case become *edge closure points* here and the closure of ∂M is denoted $\overline{\partial M}$. Some conditions have to be imposed on M and on the corresponding multi-valued radial vector field U defining a "boundary" $\{x + U : x \in M\}$ to keep control of the topology. For example locally in a neighbourhood of a singular point x_0, M may be expressed as a union of smooth 2-manifolds with boundaries and corners M_j, where two such intersect only along boundary edges, with a similar restriction for $x_0 \in \overline{\partial M}$. A general 1-dimensional skeletal set consists of a finite number of curve segments meeting only at their endpoints (for the generic Blum medial axis three segments could meet at a triple point); a general 2-dimensional skeletal set at a singular point x_0 the link L is a 1-dimensional stratified set in the 2-sphere, that is it consists of a finite number of curve segments meeting only at their ends—a skeletal set in the 2-sphere. The connected components C_i' of the complement of L in the 2-sphere correspond to the "local complementary components" C_i of $x_0 \in M$, that is the connected components of the complement $B_\varepsilon(x_0)\backslash M$ of M in a small 3-ball centred at x_0.

A radial vector field U on a skeletal set M is required to satisfy a number of conditions too. Thus at points of M_{reg} there are two *nonzro* values of U

which make dot products of opposite sign with a normal to M_{reg}, thus are "on opposite sides" of M; the behaviour has to be carefully controlled at singular and edge points as well. Jim also requires U to satisfy some *local initial conditions* which essentially limit the initial behaviour of the radial flow $x + \varepsilon U$ at the various regular, singular and edge points of M and ensure that the above complementary components of points $x_0 \in M$ are contractible.

Having set up the apparatus of skeletal sets and radial fields, it is possible to define radial shape operators S_{rad} and radial principal curvatures κ_{ri} at smooth points of M, together with corresponding concepts for points of $\overline{\partial M}$, in a similar way to §1.2.2. Then Jim finds sufficient conditions for the boundary $\mathcal{B}=\{x+U : x \in M\}$, taking all values of U here for each x, to be smooth (Theorem 2.5 of Paper 63). As with the Blum medial axis there is a 'compatibility 1-form' $\eta_U = \omega_U + dr$ where $\omega_U(v) = v \cdot U_1$, $r = ||U||, U_1 = U/r$, which is required to be zero at all points of M_{sing} including edge points. The other conditions for smoothness come from the principal (radial) curvatures: $r < \min\left\{\frac{1}{\kappa_{ri}}\right\}$ away from $\overline{\partial M}$ and a similar condition involving edge curvatures at points of $\overline{\partial M}$.

Recent extended work with Ellen Gasparovic in Papers 92, 93 and 107 has generalised medial structures to configurations of more than one object and also taking into account the relative dispositions of the various objects—the 'spaces in between'—but I shall not attempt to give an account of this work here. See §1.4.4 below.

1.2.7 "Rigidity" questions for deformations of regions and medial axes

Medial axes have been applied to comparison of shapes, though one of their drawbacks is that small perturbations of the boundary of a region can produce significant new parts to the medial axis. A number of ways have been proposed to get round this problem, often involving controlling the way in which the medial axis can evolve—see for example the discussion of M-reps in [SP], and also [YZ].

A separate issue is that even though a diffeomorphism between regions might take the one medial axis to the other, desirable properties of the medial axis might not be preserved. Some initial questions in this direction are raised in Paper 97. Suppose $\varphi : \Omega_1 \to \Omega_2$ is a diffeomorphism between two regions in $\mathbb{R}^n$ with medial axes M_1, M_2, and that (Ω_1, M_1) and (Ω_2, M_2) are homeomorphic as pairs, that is there is a homeomorphism between Ω_1 and Ω_2 which restricts to a homeomorphism between M_1 and M_2. How much more of the medial structure of Ω_1 and Ω_2 will be preserved by φ? An example given by Jim is shown in Figure 1.12; this depends on the fact that a local diffeomorphism $(\mathbb{R}^2\,\mathbf{O}) \to (\mathbb{R}^\mathbf{2}, \mathbf{O})$ has 1-jet a linear map $\mathbb{R}^2 \to \mathbb{R}^2$

and linear maps preserve the cross-ratio of four lines through the origin (§3 of Paper 97) has the necessary definition of cross-ratio and the extension to higher dimensions).

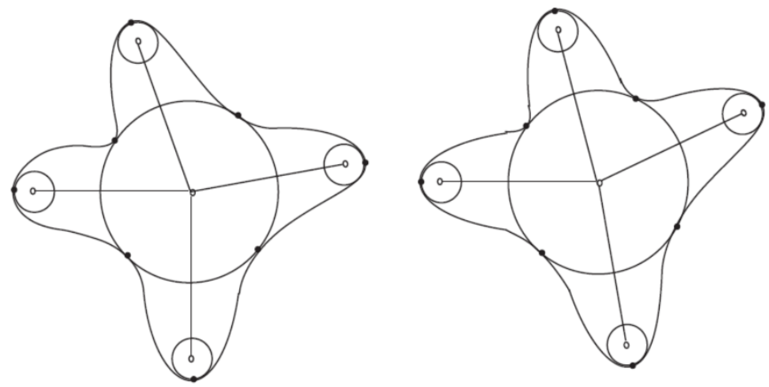


Fig. 1.12 (Figure 1 of Paper 97.) A small diffeomorphic perturbation φ takes the left-hand region to the right-hand one. It also takes the four marked curvature maxima, contact points of A_3 circles whose centres are the end-points of the medial axis on the left, to the corresponding points on the right. Likewise with the four marked contact points of the (non-generic) A_1^4 circle. But, because four branches meet in the centre of the A_1^4 circles and the cross-ratios of their tangents will in general be unequal, φ cannot be a diffeomorphism taking one medial axis to the other.

This example, which can be imitated in $\mathbb{R}^3$ with four smooth medial axis sheets, is, as mentioned non-generic, since the generic situation is for A_1^3 points in 2d or Y-junction curves in 3d, but similar problems arise with φ preserving more of the medial structure in the generic A_1^3 case. For example suppose we want to preserve the radial structure at an A_1^3 point of the medial axis, that is the three radial vectors U in addition to the three medial axis tangents. Then taking the tangents and one at a time of the radial vectors gives three sets of four lines and Damon points out that the cross ratio of each of these must be preserved by φ. The question becomes: how nearly can a diffeomorphism φ preserve important features of the medial structure at these branching points?

The above might be called preservation of first-order structure; Jim also starts to investigate second-order questions, which involve the interactions of φ with radial shape operators.

1.3 Illuminated surfaces: Peter Giblin

In an study extended over about 15 years from 2001, Jim Damon, Gareth Haslinger and I worked on the local classification of illuminated surfaces under viewer movement. (Gareth Haslinger's involvement ended in 2004.) Initially this was funded by the *Insight 2+* grant of the European Commission and subsequently partially by the Simons Foundation, NSF and DARPA among other smaller grants. Two articles directed at computer scientists (Papers 79 and 80) were followed in 2016 by a monograph (Paper 90) containing the full details of the results. I shall give here a sample of the ideas and results obtained in this project, while giving references to sources for the technical details.

The objective is to consider an object M consisting of a smooth surface, a surface with boundary, a pair of surfaces ("sheets") meeting in a crease or a triple of surfaces meeting in a corner. Given stable illumination from a single source, there will appear shade curves, where light is tangent to (one sheet of) M and cast shadows, where this light strikes (usually a different) sheet of M. In addition there may be surface markings—distinguished curves—on M and, viewed from a given direction, M will have a contour generator where viewlines are tangent to M. All these are to be viewed from a generic 2-parameter family of directions, and we seek to describe in precise mathematical terms how the various components interact at a local level. Some examples are illustrated in Figures 1.14 and 1.15 below.

Definition 1.3.1 These are the various properties of surfaces we consider:

(F) *Surface features* are boundary edges; creases (sometimes denoted Cr in diagrams) where smooth surfaces with common boundary meet; corners where three surfaces with boundary meet in three creases; and surface markings, that is curves in a fixed position on any of the above. The various "visually different" types of creases and corners are illustrated in Figure 1.16.

(S) *Shade/shadow* A shade curve (S) is the contour generator of one sheet of *M from the direction of the incident light*: light rays are tangent to the surface along this curve. It is also called a "terminator": the observed brightness of an illuminated surface gradually decreases towards this curve and beyond it there is no illumination. Thus the shade curve is not in practice sharply delineated, but here we treat it as an exact curve on a surface. A cast shadow (SS) is a curve on one sheet of M along which light rays forming a shade curve further strike the surface. Since we consider fixed stable light projections the shade/shadow curves are *fixed* on the surface in the same way as surface features.

(C) *Contour (generators)* are curves on a surface where lines *in the viewing direction* are tangent to the surface. (The corresponding curve in a "viewing plane" would be an apparent contour (or profile/outline); we shall use the word "contour" for the curve on the surface.) Contours will change as the viewpoint changes unless the view projection is stable.

See Figure 1.13 for a real-world example. Interactions between these ingredients can be denoted by (SF), (SC), (FC), (SFC).

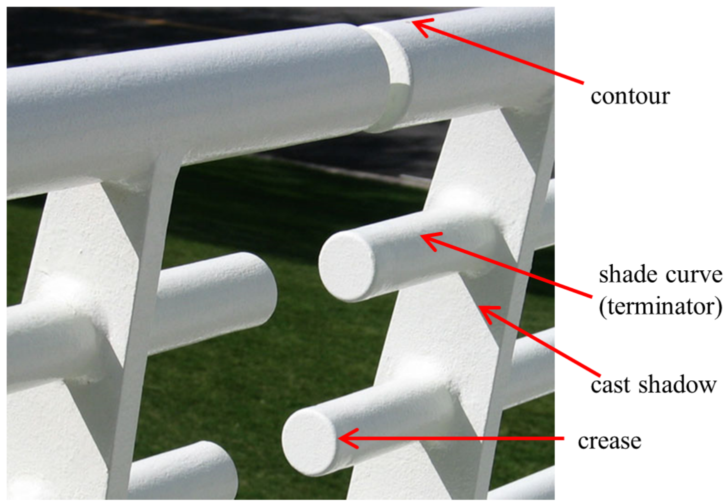


Fig. 1.13 A metal railing illustrating Definition 1.3.1. (Photo by Jim Damon.)

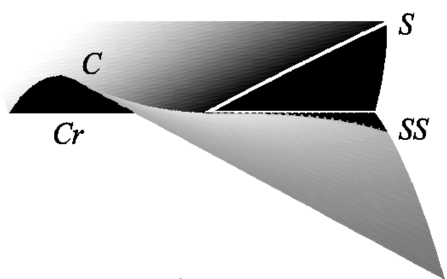


Fig. 1.14 A "crease", that is two smooth surfaces meeting along a common boundary Cr, with illumination from upper left producing a gradual shading with shade curve S as the "terminator" where incident rays are tangent to one sheet of the crease. These same rays cast a shadow SS on the other sheet, and light rays from the surface in the direction of the observer are tangent along C, the contour (generator), which here terminates at the crease. This is an interaction between a feature (the crease), shade/shadow and contour, so is an example of SFC. As the viewer moves the interaction between the three will evolve.

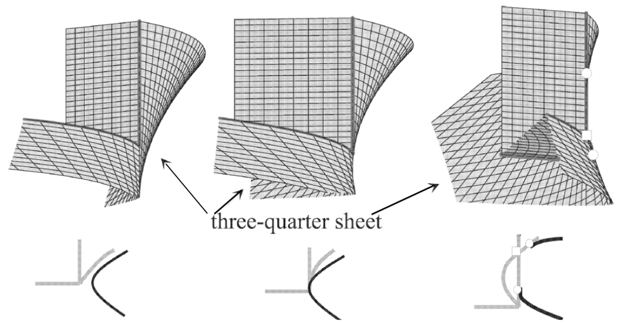


Fig. 1.15 A corner exhibiting only an evolution of the apparent contour as the viewpoint changes. The lower diagram is a schematic representation of the three creases making up the corner. (Adapted from Figure 12.11 in Paper 90.) "Three-quarter sheet" refers to the sheet making up the corner which contains an angle greater than 90° between creases. Here we are assuming that the stable light projection is a submersion on each of the sheets, so that there are no shade curves or shadows. It is an example of FC, an interaction of features (creases) and contours which evolves as shown under generic viewer movement.

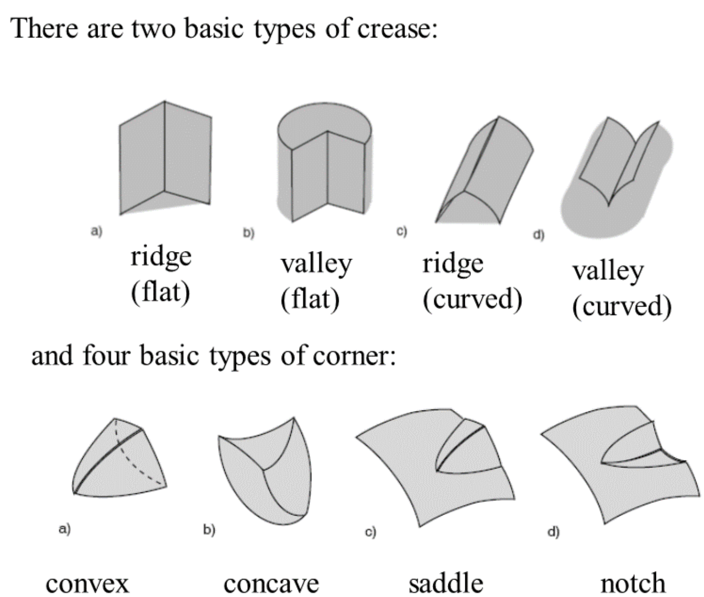


Fig. 1.16 The "visually different" types of crease and corner. The choice among these affects for example how it is physically possible to form cast shadows. (Adapted from Figs.2.2 and 2.4 of Paper 90.)

1.3.1 The mathematical setup

Consider a surface germ $(M, \mathbf{p})$ together with two directions, an incident light and a view. These are always assumed to be distinct. Let φ project towards the light source and ψ project towards the viewer:

$$\big(\mathbb{R}^2, \varphi(\mathbf{p})\big) \xleftarrow{\ \varphi\ } (M, \mathbf{p}) \xrightarrow{\ \psi\ } \big(\mathbb{R}^2, \psi(\mathbf{p})\big) \tag{1.9}$$

Note that a cast shadow can be regarded as a point of $\varphi^{-1}\varphi(C)$ where C is the critical set of φ.

We do not consider equivalence of divergent diagrams (1.9) since such equivalence lacks the necessary versal unfolding and finite determinacy theorems (see p.42 of Paper 90); rather, we assume that the "light projection" φ is *stable*, that is stable under small perturbations preserving the features (F). For example the projection φ should not be along a direction tangent to a crease or to a surface marking. From this we deduce the stratifications consisting of the features and shade/shadow curves (S): there may be more than one configuration of shade/shadow curves for stable light projections. The final step is twofold: we apply singularity theory, in particular Jim's theory of "geometric subgroups of the right-left group $\mathcal{A}$" (Paper 24) to classify abstract viewing projection germs ψ, and we check to find whether all the abstract germs can be realised by actual surfaces. This final classification is carried out by an equivalence preserving a stratification of the source M according to the configuration of S, F on M. The classification is done up to codimension 2, sometimes passing to topological equivalence since there are situations where smooth equivalence produces moduli (such as four curves through a point having a cross-ratio of tangent directions). Naturally this whole process is lengthy and it is spelled out in Paper 90. Here, I shall concentrate on the general ideas and some examples.

1.3.2 Examples

Example 1.3.2 C^1 parabola

Consider the surface $M : z = x^3 + xy$ with light source along the positive x-axis, that is light projection $(x, y, z) \to (y, z)$, so that $\varphi : (x, y) \to (y, x^3 + xy)$ in terms of a local parametrization of M. The critical set is $y = -3x^2$ which is a parametrization of the shade curve: it consists of points $(x, -3x^2, -2x^3) \in M$. The image of this curve under the light projection is a cusp $(-3x^2, -2x^2)$ so this is a *cusp light* situation. The cast shadow curve consists of those points of M where the light rays through the shade curve meet M again. This can be calculated by considering these rays $(x + \lambda, -3x^2, -2x^3)$, which meet M again where $\lambda = -3x$, resulting in the shade curve $(-2x, -3x^2, -2x^3) =$

$(u, -\frac{3}{4}u^2, \frac{1}{4}u^3)$ say, that is the curve $y = -\frac{3}{4}x^2$ on M. However it is more straightforward to use the fact that the cast shadow curve and shade curve together are the inverse image on M (under the light projection) of the critical values (discriminant) of the light projection, that is of the set $(-3x^2, -2x^3)$. Those points of the form $(u, v, u^3 + uv)$ projecting to $(-3x^2, -2x^3)$ are easily checked to have $u = x$ (the shade curve) or $u = -2x$, which is therefore the cast shadow as above.

But there is an additional consideration: only half the shade and half the cast shadow will be physically present. This is because half of the theoretical shade curve and half of the cast shadow are locally occluded by the surface M itself. Together the two physically present halves make up a "C^1 parabola": a curve with a continuous tangent but discontinuous curvature. See Figure 1.17, left, which shows a generic view which is stable. A mathematical problem when finding normal forms is then to use an equivalence which preserves this non-smooth curve on M. This is a codimension 1 situation: from a generic viewpoint in the tangent plane to M at the origin, for example along the y-axis, the view projection is a fold mapping and a single direction of movement out of the tangent plane unfolds the view to a generic one. An abstract rendering of this, not taking account of visibility, is in the lower part of Figure 1.18, the evolution with a single parameter λ. This is a "cusp light, fold view" situation. Note that it is an example in which a smooth surface casts a shadow on *itself* rather than on another surface.

□

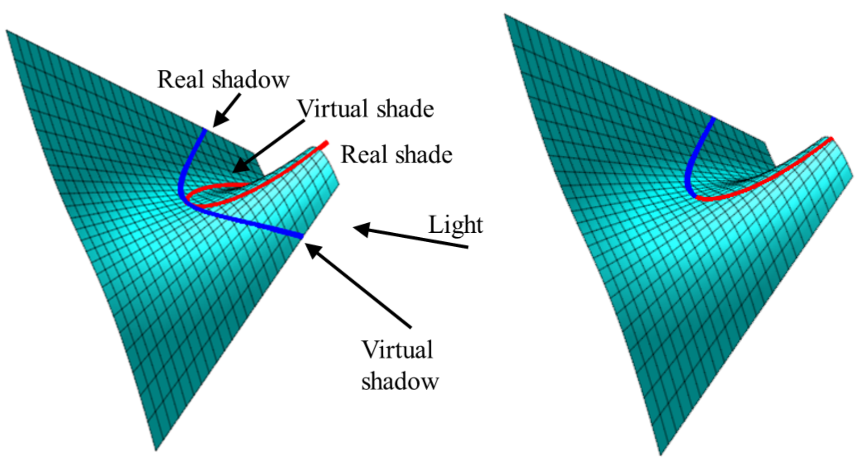


Fig. 1.17 Left: the full calculated shade and shadow curves in Example 1.3.2, viewed for clarity from a *generic direction*, not in the tangent plane to the surface at the origin. The light projection has a cusp singularity. Right: the half of each of these curves which is real, that is physically generated on the surface by light from the right of the picture. The light projection will line up the two so that the shadow is directly behind the light.

Example 1.3.3 "Cusp light cusp view"

Consider instead M given by $z = xy+x^3+y^3$ so that the asymptotic directions at the origin are still along the coordinate axes, the projections along these axes will now *both* create *ordinary cusps* in the image: taking light along the y-axis and view along the x-axis gives a "cusp-light, cusp-view" codimension 2 situation. See Figure 1.18, bottom, the "clock diagram" with λ, μ axes.

Note that the shade and cast shadow curves are regarded as *fixed* curves on the surface, with a stable light direction, and it is the view which changes in a 1- or a 2-parameter family. The special thing about shade and cast shadow curves is that, unlike say a surface marking which would be assumed smooth, or terminating at a crease say, they together form a C^1 curve on the surface. □

1.3.3 Notes on Examples 1.3.2 and 1.3.3

The abstract classification of view projections proceeds by starting with a (local) model surface, in this case a plane $\mathbb{R}^2$. For a crease, such as Figure 1.14, the model would be two coordinate planes in $\mathbb{R}^3$ meeting along a coordinate axis and for a corner, such as Figure 1.15, it would be the three coordinate planes in $\mathbb{R}^3$. On this surface we take models for the fixed shade/shadow curves: in the present examples we can use the semi-analytic set $\mathcal{V} = \{(x,0) : x \leq 0\} \cup \{(x,x^2) : x \geq 0\}$ which is contained in the analytic set $\{(x,y) : y(y-x^2) = 0\}$, that is an axis and a parabola. Note that this equation is weighted homogeneous with (x,y) having weights $(1,2)$. It is the set $\mathcal{V}$ which must be preserved by changes of coordinates when reducing the view projection to a normal form, with appropriate unfolding terms corresponding to its ${}_{\mathcal{V}}\mathcal{A}_e$ codimension. According to results contained in Chapters 5 and 6 of Paper 90, what Jim calls *special semi-analytic sets*, of which $\mathcal{V}$ and the others occurring in the current investigation are examples, can be handled for this classification in the same way as their analytic completions. In the present case the module of analytic vector fields $\mathrm{Derlog}^{an}(\mathcal{V})$ is generated by two vector fields: the Euler vector field $x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ defined by the weights, and the Hamiltonian vector field $(2y - x^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial dy}$. Thus the Euler field and $y\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$ can be used as a basis. Classification up to ${}_{\mathcal{V}}\mathcal{A}$- equivalence can now proceed along fairly standard lines, using complete transversals [BKD]. In practice the authors of Papers 79, 80 and 90 used computer assistance, in particular the package TRANSVERSAL [Kir] written for Maple™ by Neil Kirk, to perform most of the very large number of complete transversal and jet sufficiency calculations. See Chapter 6 of Paper 90 for further details of this.

In Example 1.3.2 a normal form and versal unfolding for the view projection f come to $f(x,y) = (x-y, xy+y^2+\lambda x)$ and in Example 1.3.3 to $f(x,y) = (x-y, xy-y^2+x^3+\lambda x+\mu x^2)$. These are the source of the unfolding diagrams in the lower part of Figure 1.18.

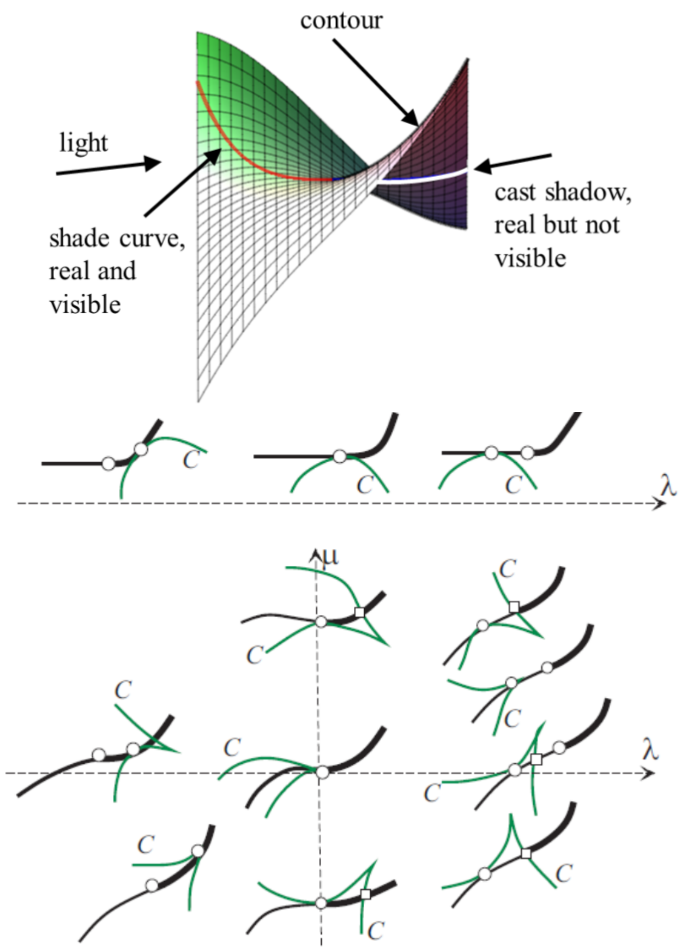


Fig. 1.18 Above: Example 1.3.3, $z = xy + x^3 + y^3$, light projection (from the direction $(0, 1, 0)$) is a cusp singularity and the view projection (direction $(1, 0, 0)$) also, giving a cusp contour, half of which is visible. The cast shadow is real but from the view shown cannot be seen, since occluded by the surface. while the shade curve is both real and visible. As the view changes, pieces of the cast shadow become visible. Below: An abstract rendering of (upper diagram) the cusp light fold view of Example 1.3.2, as unfolded by a single unfolding parameter λ varying the view; and (lower diagram) cusp light cusp view of Example 1.3.3, as unfolded by two independent parameters λ, μ varying the view, in a "clock diagram". The light coloured curve C is the contour, the thin and thick dark curves are the shade and cast shadow curves, and this diagram takes account of the physical existence of shade and shadow curves, but does not take account of visibility. (Adapted from Figure 6.2 of Paper 90.)

1.3.4 Comments on enumerating the various cases

There is a total of 14 abstract models for illuminated surfaces in Figure 6.1 of Paper 90: for example the situation in Figure 1.14 is modelled locally by the planes $x = 0$ and $y = 0$ with a line (modelling the shade curve) *transverse* to the z-axis lying in one plane and a curve (modelling the cast shadow) *tangent* to the z-axis lying in the other plane. (The cast shadow of one sheet of a crease on the other sheet will always be tangent to the intersection line of the two sheets.) Thus the semi-analytic set consisting of two lines and a curve are to be preserved when reducing the view projection to a normal form. Note that the critical set of this view projection, that is the contour, can lie on either sheet. During this reduction there appear moduli whose values affect the topological nature of the images. Jim provided in Chapter 7 of Paper 90 the necessary methods to justify topological versality, that is roughly speaking to show that "all possible visual interactions"' between features, shade/shadow and contours are captured by the diagrams displayed. The moduli space (either 1 or 2 dimensions) is partitioned into regions in each of which the geometrical relationship between shade, features and contour remains qualitatively constant.

1.3.5 The further role of geometry

Geometrical considerations arise in another way besides that sketched above: each abstract normal form and its unfolding needs to be realised as a smooth surface, surface with boundary, crease or corner with appropriate features. In some cases this turns out not to be possible, and here I shall give one example. The full treatment of this topic is in Chapter 9 of Paper 90.

The abstract classification of projections of surfaces with boundary ([BG, Gor]) amounts to a classification up to ${}_{\mathcal{V}}\mathcal{A}$-equivalence of map germs $\mathbb{R}^2 \to \mathbb{R}^2$, where $\mathcal{V}$ represents the boundary, so can be taken as the x-axis. This means standard left-right or $\mathcal{A}$ equivalence but with the change of coordinates in the source preserving the set V. (See §1.1.2.) It is crucial that all the sets V encountered in this work make this group of V-preserving diffeomorphisms into a geometric subgroup of the $\mathcal{A}$ group. See p.49 of Paper 90.

In the context of illuminated surfaces $\mathcal{V}$ can be taken instead to represent a shade curve on a smooth surface (or a marking curve). Thus existing classifications feed into the present investigation in natural way (the same applies for instance to classifications such as those of Tari [Tar]). In the case of a shade curve, $\mathcal{V}$ is not arbitrary but the critical set of a projection, and therefore it might be expected that not all cases will transfer to this context.

A well-known theorem states that the tangent direction of projection and the tangent to the resulting contour at a point $\mathbf{p} \in M$, this being a smooth surface, are conjugate with respect to the second fundamental form of M at $\mathbf{p}$. Taking the projection direction as the *light* direction the contour is the shade curve S and if this is viewed in the conjugate direction, that is along a tangent to S, the image, that is the view of the shade curve, will therefore in general be a cusp. (Since conjugacy is symmetric viewing C in the light direction will also show a cusp.) See Figure 1.19

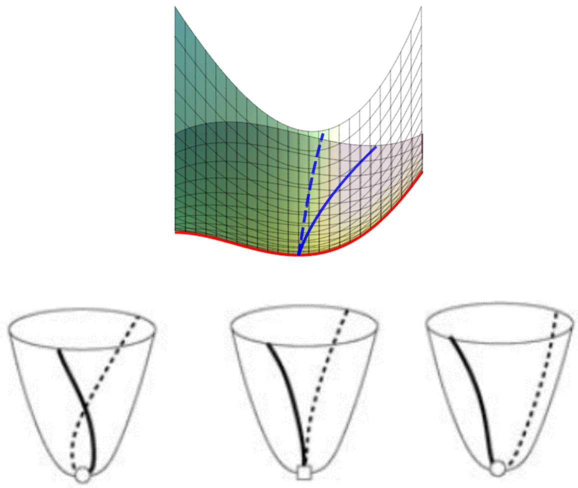


Fig. 1.19 Top: a smooth surface, shown semi-transparent to see the shade curve forming a cusp, with the far branch dashed and not visible from this side. The view direction is tangent to the shade curve. In this diagram light and view directions are along the y- and x-axes; to be conjugate the term in xy is absent from the Monge form at the origin. Further generic conditions on the quadratic and cubic terms ensure that the view of this shade curve is an ordinary cusp. Bottom: a schematic diagram of how this singularity unfolds with viewer movement.

Could both the contour C and the image of S in the view direction exhibit a cusp at $\mathbf{p}$? This is called a *double cusp* in the context of projections of surfaces with boundary, where it has codimension 3 with a smooth modulus and therefore requires topological ${}_{\mathcal{V}}\mathcal{A}$ equivalence. But in the present context it cannot occur at all. To see this, if both the shade curve and the contour project to cusps in the view direction $\mathbf{V}$, then $\mathbf{V}$ must be tangent to both critical sets. In the case of the contour this means that $\mathbf{V}$ is in an asymptotic (i.e. self-conjugate) direction at $\mathbf{p}$, and in the case of the shade curve it means that this self-conjugate direction is also conjugate to the light direction $\mathbf{L} \neq \mathbf{V}$, by the conjugacy result above. This can only be so at a parabolic point $\mathbf{p}$, but viewing a parabolic point along its unique asymptotic direction does not produce a cusp: the contour is then itself singular and the projection is generically an isolated point or a crossing.

To create a double cusp using a marking curve it is only necessary to have the view direction tangent to the contour, that is in an asymptotic direction at $\mathbf{p}$ and to have the marking curve (or boundary edge of the surface) tangent to the contour; see Figure 1.20.

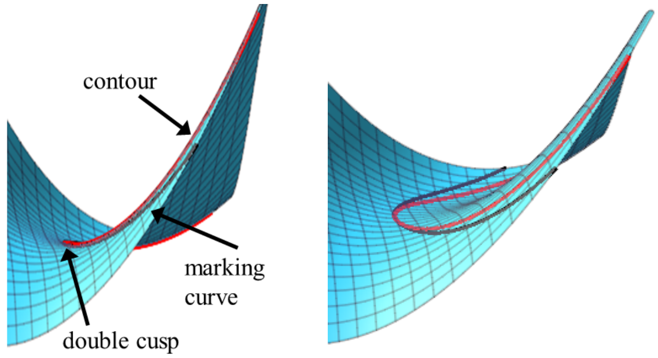


Fig. 1.20 Left: a double cusp formed by a marking curve (darker curve) and a contour (lighter curve); right: the view direction perturbed to show these curves are tangent on the surface.

A more subtle restriction imposed by the present context concerns the "semi-goose" singularity of [BG, Gor], which, as a map $\mathbb{R}^2 \to \mathbb{R}^2$ under ${}_{\mathcal{V}}\mathcal{A}$ equivalence where $\mathcal{V}$ is the y-axis, has normal form $(y, x^2 + xy^3)$ and codimension 2. In this case a stable light projection map can result in a semi-goose view projection where the shade curve takes the place of the boundary, but viewer movement cannot realise a versal unfolding of the semi-goose singularity (Theorem 9.3(iv) of Paper 90). This is because a singularity which for the abstract map is in a neighbourhood of the semi-goose ("semi-lips/semi-beaks") cannot occur through viewer movement.

1.3.6 Techniques needed for the classification

It is not possible to give here an exposition of all the techniques used to carry through the classifications of Papers 79, 80 and 90, but here is a general list to emphasize the heavy dependence on Jim's technical theorems.

1. Jim Damon's theory of geometric subgroups of $\mathcal{A}$ and $\mathcal{K}$ (Paper 24). (For more information on this see §1.1.) This is because abstract classifications of view projections are required to preserve configurations of shade/shadow curves, pairs or triples of lines where two or three surfaces meet in a crease or a corner, boundary edges and lines representing surface markings. These together form "special semi-analytic sets" in $\mathbb{R}^2$ or $\mathbb{R}^3$ and the extension of results from Paper 24 to cover such sets is given in Chapter 5 of Paper 90.
2. Jim Damon's theory of topological triviality and versality for subgroups of $\mathcal{A}$ and $\mathcal{K}$ (Paper 30). This is because the abstract classifications sometimes contain smooth moduli and the nearest formal equivalent to the concept

of “visual equivalence” is captured by smooth equivalence outside a single basepoint and overall topological equivalence. The necessary results for the application to the current context are given in Chapter 7 of Paper 90.

3. Methods for carrying out the classification using complete transversals [BKD] and sufficiency of jets. In practice the large number of calculations requires computer assistance using for example the software TRANSVERSAL [Kir] and the above theoretical results.
4. Techniques for realizing abstract germs as projections of actual surfaces with corresponding features and shade/shadow curves. The techniques for this are given in Chapter 9 of Paper 90.

1.4 Applications to medical image computing: Stephen Pizer

1.4.1 Introduction

From approximately the year 2000 through his death in 2022, Jim Damon, Professor of Mathematics at the University of North Carolina (UNC), was an active research collaborator with Professor Stephen Pizer in the research group led by Steve centered in the UNC Department of Computer Science. The collaboration was in the area of medical image computing. Jim often stated that he was a pure mathematician, yet he enjoyed developing his mathematics, stimulated by the medical image computing problems from the research of Steve’s group. His contributions also involved active co-advising of Computer Science doctoral students’ PhD dissertation research that had Steve as principal advisor.

Many research objectives of Steve’s group were the target of Jim’s singularity theory contributions. The stimulation came in two forms: 1) Most often, Jim would attend weekly meetings of the research group and hear about objectives that he judged could be helped by his mathematics; or 2) Steve would ask Jim about the possibility he could work on some particular mathematical challenge relevant to his medical image analysis objectives. Then months would pass without a response from Jim. Again and again, after that interval, Jim would provide some mathematics to Steve, with the comment “Perhaps this might be useful to you.” Indeed, the usefulness was never in question; it was always very helpful! And the mathematical depth and generality was great. I shall present here the medical image computing challenges that were provided to Jim and an overview of the mathematics he produced in response.

The part of Steve’s research on which Jim targeted his contributions focused on the *statistics of the shape of anatomic objects*. Most of that research involved collaborations with UNC Statistics Professor, J.S. (also Steve) Marron, and some of Jim contributions were with Steve Marron alone. The objectives of Steve Pizer's research were 1) to extract ("segment") objects from images to allow planning of treatments such as radiotherapy; 2) to diagnose disease based on object shape; and 3) to determine what the shape differences were between diseased and normal anatomic objects.

1.4.2 Segmentation and height ridges

In images, and indeed in any measured entity or application of a physical or computed operator, a spatial aperture is necessarily used as a weighting function about the point at which the measurement is centered. This is true even of operators such as derivatives, which in the ideal sense but not in any computationally applicable sense, have infinitesimal aperture. The width of the aperture is called its spatial scale. This width is typically measured by the square Root of the Mean of the Square (called the RMS width σ) of the distance multiplied by the aperture value at that distance. The functional form of the aperture is chosen to achieve a variety of properties: necessarily not creating structure, i.e., upon application decreasing the amplitude of relative maxima and minima of the operand; being equivariant over rotation, translation, and magnification; forming a semi-group over σ; and having a finite integral of its square. The isotropic, zero-centered Gaussian $g(x, \sigma)$ is provably the unique aperture form that satisfies that requirement [tHR].

The appropriate spatial scale to use on an image or object depends on the distances being sensed and the signal-to-noise level in the entity being operated upon. And different scales are appropriate at different locations and for different measurements. Thus, Koenderink [KvD] and others saw an image $I(x)$ with $x \in \mathbb{R}^n$ in the form of a continuous stack over the spatial scale σ, i.e., a function on the "scale space" $(x, \sigma) \in \mathbb{R}^{n+1}$: $J(x, \sigma) = g(x, \sigma) * I(x)$, the asterisk denoting convolution. Segmentation, involving locating an object in an image (Paper 60), and statistics [CM], involving operations such as classification and hypothesis testing on measurement data, were shown to be effective in this scale space.

One form of segmentation studied in Steve's group involved what Jim and Steve called *height ridges* (there is also some discussion of these in §1.2.5). See Figure 1.21. Object boundaries were seen as height ridges of the gradient magnitude of an image $I(x)$, and bar-like objects were seen as height ridges of $I(x, w)$, where w was seen as local object width. Mimicking how human vision works, the object width was dealt with at scale σ, proportional to width. Jim started creating mathematics focusing on height ridges as functions of x, or (x, σ), or (x, σ, u) with u a unit vector in $\mathbb{R}^2$, or (x, σ, u_1, u_2) with u_1, u_2 orthogonal unit vectors in $\mathbb{R}^3$. This height ridge mathematics will now be described mathematically.

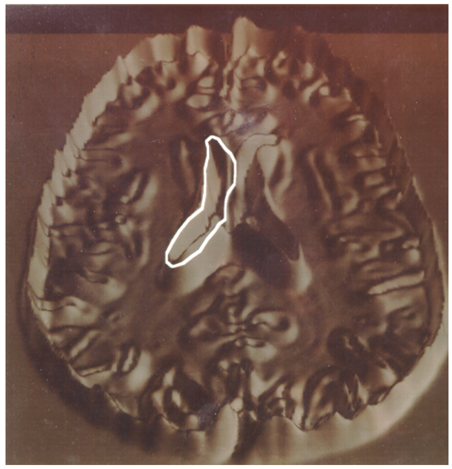


Fig. 1.21 Gradient magnitudes from a slice from a magnetic resonance image of the brain. The ridges show boundaries of brain and head structures.

As just motivated, many loci in images can be found as height ridges of a scalar function F, defined on a space of some dimension, this function being derived from an image I in $\mathbb{R}^2$ or $\mathbb{R}^3$. For example, F could give the strength of some property such as contrast. A d-dimensional height ridge of a function of n variables is a type of extremum in an appropriately chosen $n-d$ dimensional subspace. For example, if $I(x, y)$ is the image and $F(x, y, u)$ is the directional derivative of I at (x, y) in the u direction, where u is a unit vector in $\mathbb{R}^2$, then a boundary of good contrast of an object can be understood as the 1-dimensional ($d = 1$) height ridge of F, defined on $\mathbb{R}^3$.

Let the object have the property that I in its near-boundary interior is higher than that in its near-boundary exterior. A common choice of the $n-d$ dimensional subspace is that spanned by the eigenvectors of the Hessian D^2F whose eigenvalues, for our F, are the two most negative (the ridge is convex) of D^2F when it has at least two negative eigenvalues; we called such ridges "maximum convexity" height ridges. No point of this F is a height ridge point if it does not have at least two negative eigenvalues.

The question Steve asked Jim was whether height ridges of an F could cross and whether they could stop. Damon's mathematical investigation proved that being zeros, i.e., level surfaces of derivatives of F, with fully determined Hessian eigenvalues at each point, the ridges could not cross and that they could end only by the sign of one of the eigenvalues of F crossing zero from negative to positive. He therefore defined a generalization of height ridges to *types*, with the type determined by the number of positive eigenvalues at a point on the locus, and he called the non-ridge types "connection curves" because, he proved, the eigenvalue sign change that caused a ridge to end could later cross zero back if it was followed along the subdimensional saddle of F.

Many thought that the tangent to the height ridge locus would be orthogonal to the extremum-direction-determining F eigendirections, but Jim showed that this only occurred with singular conditions of symmetry of the two sides of the ridge occurring at the ridge.

Jim's results became even more important when the object boundary locus in $I(x)$ with $x \in \mathbb{R}^n$ was to be found as an $n-1$ dimensional locus in the scale space $\mathbb{R}^{n+1}$. Now, the ridge has codimension 2. Damon showed that his mathematics, using maximum convexity ridges, applied not only to ridges of codimension 1 but also to ridges of higher codimension, here 2.

Steve had found that object width at locations along all but its shortest axis was important for both object segmentation and shape-based diagnosis. One direction he explored found a height ridge locus of a medial strength function which in 2-dimensional images is $F(x, \sigma, u, \theta)$, with x being medial with respect to the boundary (see §1.2 for more information on medial structures), σ being proportional to the object width at medial point x, u being orthogonal to the putative medial locus (= medial axis), and θ being the angle from u at which image contrast is measured.

This height ridge was called a core. When $x \in \mathbb{R}^2$ so $(x, \sigma, u, \theta) \in \mathbb{R}^5$, the dimension d of the ridge was 3, giving (x, σ) and $n - d = 2$. When $x \in \mathbb{R}^3$ so $(x, \sigma) \in \mathbb{R}^4$, there is an additional angle from u to determine the direction in which to measure image contrast, so the dimension d of the ridge was 4 and $n = 7$ and thus $d - n = 3$. Again, Damon's findings on maximum convexity ridges, that his mathematics applied not only to ridges of codimension 1 but also to ridges of higher codimension, here 2 for 2-dimensional images and 3 for 3-dimensional images, was important.

Some of the height ridges from image analysis choose the extremum-based directions by a condition other than eigenvectors of the Hessian. For example, for the "Canny ridge" to determine an object boundary of a 2-dimensional or 3-dimensional image I [Can], the function F is the gradient magnitude of the image I and the direction of subdimensional maximization is the gradient direction. In Pizer's cores, also called the multiscale medial axis [Fr], one choice of one of the directions of subdimensional maximization was the unit vector in the width-proportional σ and another could be the direction orthogonal to the ridge direction u in 2-dimensional images or the ridge-spanning directions, u_1 and u_2 in 3-dimensional images. The mathematical question arose as to whether Damon's mathematics applied fully when these directions of subdimensional maximization were not maximum convexity directions. Damon's mathematical study showed that his previous results did not apply.

In image processing algorithms a common segmentation approach involved finding ridges as satisfying the watershed property, related to catchment areas of flow along the negative gradient of the function F derived from the image. The mathematical question arose of whether these ridges were generically height ridges. Discussions between Jim and Steve determined that they could not be because the height ridge property is locally determined whereas the watershed ridge property is determined at a distance; that is, changes in F far from the putative ridge point could determine whether that point was a watershed ridge or not.

1.4.3 Skeletal models

Note The terms "medial locus" here and "medial axis" in §1.2 are synonymous.

When Steve's attention turned from applications of image intensity patterns to statistics on object models, he realized that the properties of object width and curvature of the interior of an object could be important geometric features on which to do statistics (and so it has turned out!). Steve realized that the medial models of Blum [B] (see also §1.2), derived by eroding measured object boundaries, were designed to provide such features. Blum had understood these models as coming from the set of spheres entirely within the object (Figure 1.22) and bitangent to its boundary and thus yielding a

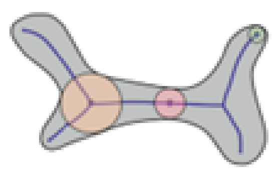


Fig. 1.22 A 2-dimensional object showing three bitangent circles: one at a branch point (tritangent), one at a normal point, and one at an endpoint (higher order tangency) [YZ]. In 3-dimensions it is spheres that are bitangent.

sphere center and radius for each sphere, as well as derivatives of these values along the medial locus. However, Steve, wanting to use discrete samples of the properties, considered the local width information to be carried not by the radius of a sphere but by the vectors from its center to the points of boundary intersection; he called these vectors "spokes". See Figure 1.23.

Steve realized that the Blum method failed to yield useful shape statistics because the noise in the boundary measurements led to bushy medial locus trees with variable branching patterns across training cases. Thus, he devised a reversal of Blum's approach whereby instead of the boundary implying the medial locus and its associated spoke vectors, the medial locus implied the boundary. Then an optimization over the medial positions and the associated spokes could provide the medial locus whose implied boundary best fit the target boundary of a training instance. This allowed the branching pattern

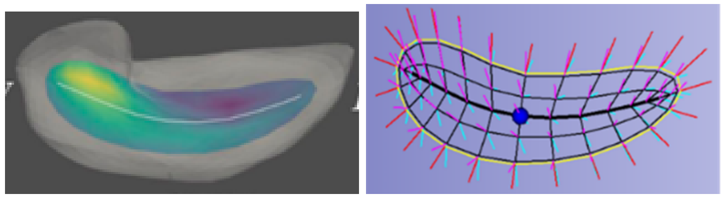


Fig. 1.23 Left: A hippocampus with its skeleton (colors show skeletal widths). Right: the spatially sampled skeleton bounded by its fold (yellow), the skeleton's skeleton ("spine", in blue) and its center (bullet), and the spokes from the skeleton to the boundary (red). (Paper 101)

of the medial locus to be fixed and thus to allow statistical analysis without needing statistics on tree patterns.

Jim picked up powerfully on this mapping from a medial locus with associated spokes to the boundary. Focusing especially on objects in 3-dimensions surfaces with spherical topology, for which the medial locus is generically a double copy of a bounded curved surface, possibly with branching singularities, his surprising mathematics first generated theorems on the way the spoke directions swing with respect to motion on the medial locus. His analysis yielded a shape operator S_{rad} (see also Definition 1.2.4 and §1.2.6) which gave these derivatives after projection onto the medial locus using the mostly non-orthogonal coordinate system of a frame on the medial locus and the spoke direction. His mathematics also generated a shape operator S_E (see Definition 1.2.6) which gave these derivatives at the edge of the medial locus, where the spoke is in the limiting tangent plane of the medial locus and ends at a crest point (ridge point) on the object boundary. Finally, he noted the importance of the fraction τ (he called it t) of the spoke length as a "radial distance". Level surfaces of the radial distance are what Jim and Steve called "onionskins". See Figure 1.24.

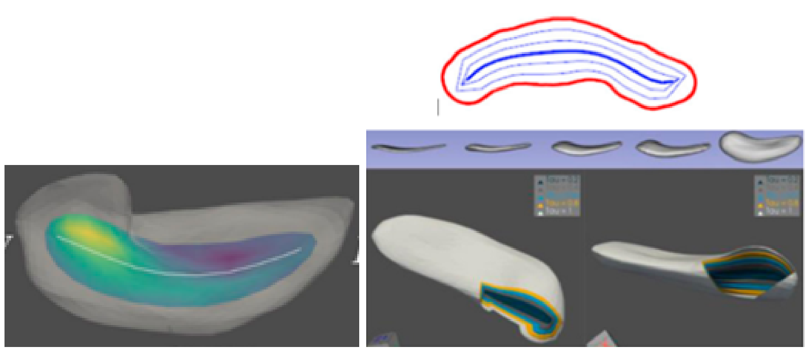


Fig. 1.24 Skeletal onionskins. Top: in 2-dimensions. Bottom: a hippocampus in 3-dimensions. (Paper 101)

By singularity theoretic analysis Jim proved that the eigenvalues of the S_{rad} operator as compared to the spoke lengths determine whether there is a singularity related to the illegality of two spokes crossing. This was important in fitting the models to a boundary because the fitted model needed to avoid such illegalities. He also showed that the ordinary shape properties describing curvature of normals on all onionskin surfaces (including the medial surface ($\tau = 0$) and the boundary implied by the spoke ends ($\tau = 1$) is completely determined by the value of what Jim called the "principal radial curvatures", which are the eigenvalues of the 2×2 matrix S_{rad} (see 1.2.5). It followed, he showed, that the signs of the principal curvatures of the ordinary shape operator and those of the principal radial curvatures were the same. The result was that curvature region types of being convex, hyperbolic, concave, and parabolic were shared across all of the onionskins.

Jim also showed how integrals on the object interior or on the object boundary can be reduced to integrals on the medial locus (Paper 71).

Fitting the model to a boundary required interpolation of the discretely sampled medial description into a continuous medial descriptor. Jim studied how to do interpolations of the medial locus and especially of the spokes from the discrete samples, all according to the radial shape operator and its eigenvectors and eigenvalues.

A property of medial models that Jim pioneered was the study, using cross ratios, of rigidities of how they varied with diffeomorphisms of the space containing the objects (Paper 97); see also §1.2.7. Certain algorithms for computing a medial model depended on applying a diffeomorphism to a base model, such as those of Yushkevich et al. [YZ] (a dissertation advisee of Steve to whose dissertation research Jim contributed; see No.7 in §1.6.2). Jim's work not only provided a base from which further work in mathematics on the effect of diffeomorphisms on medial properties has proceeded, but in the year before his death he wrote to Yushkevich, now Professor at the University of Pennsylvania, counseling him on some of the mathematical assumptions that he was apparently making.

Later, Jim and Steve realized that better fits to the boundary could be obtained by weakening to soft penalties the medial constraints of equal lengths of the spokes emanating from a fixed position on the medial locus, orthogonality of the spokes to the boundary where they end, and the behavior of the edge spokes. Jim showed that these skeletal shape descriptors, which Steve called "s-reps", followed most of the properties he had proved for fully medial descriptors and that the remainder followed the condition held that he called "partial Blum", namely that the spokes were orthogonal to the locus of the spoke ends. (There is some more information on skeletal structures in §1.2.6). Note that there is a similar concept of M-rep developed in No.7 of §1.6.2 by Paul Yushkevich and related specifically to medial structures. See also [SP, YZ].

Steve suggested that a smoothing flow of the boundary of any object to an ellipsoid and a reverse flow carrying the ellipsoid's analytically known medial model skeleton could yield a means of producing skeletal models with strong positional correspondence and thus support shape statistics strongly. Jim pointed out that the transformation between ellipsoids by the ratios of their principal radii did not maintain their medial skeletons. He went on to do the mathematics yielding a diffeomorphism between ellipsoids that did maintain those skeletons.

Jim questioned the idea that Pizer and Liu's diffeomorphism of the ellipsoid (Paper 99) strongly enough maintained skeletally-based correspondence within a set of training cases for statistics, and especially Steve's idea that fitted frames in the closure of the interior of an ellipsoid, carried by this diffeomorphism to the target object could provide important geometric features of that objects. Jim stated that the desired behavior should be skeletally intrinsic: it should make the within-object diffeomorphism between the ellipsoid and the target object maintain the skeletal properties all through the application diffeomorphism. For example, the spokes in the ellipsoid should stay straight and keep correspondences of radial distances through the application of the diffeomorphism. In regard to the desired diffeomorphism, in his last months Jim produced a preprint (Paper 97) on restrictions on producing such a diffeomorphism in the Blum situation when the axis branched. Jim passed away before implementation of this idea of diffeomorphisms maintaining a skeleton and its spokes on a non-branching skeleton, but that implementation is now being programmed, and that idea is being evaluated according to the improvement it provides for statistical classification on autism by hippocampal shape.

Jim noted that the fitted frames to the closure of the ellipsoid could be understood to transform in an affine fashion through the diffeomorphism and that the lengths and angles of the frame vectors at selected localities within the target object could be geometric features that were useful in statistical classification. Liu (Paper 103) showed this indeed to be the case.

1.4.4 Multiple objects

While single object representation and statistics had been studied, Steve noted that inter-object geometric relationships could be important. Jim and his mathematics advisee Ellen Gasparovic thence studied the medial singularities of the space between objects. Their mathematics involved extending the objects' spokes to an exterior "linking" medial surface (Papers 92 and 93). See Figure 1.25. Picking up on aspects of that work, Zhiyuan Liu, advised by Steve and Jim, developed a method of finding a linking surface approximating the Damon inter-object medial structure and the spokes to it in a way that avoided spoke crossing . Liu applied this description between a pair of brain structures, and he showed that using the link features concatenated to

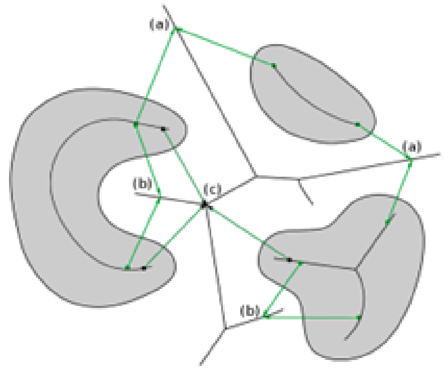


Fig. 1.25 Inter-medial-axis links between 2-dimensional objects (No. 2 in §1.6.2)

the object s-reps features produced better classification than using the object features alone (Paper 103). See Figure 1.26.

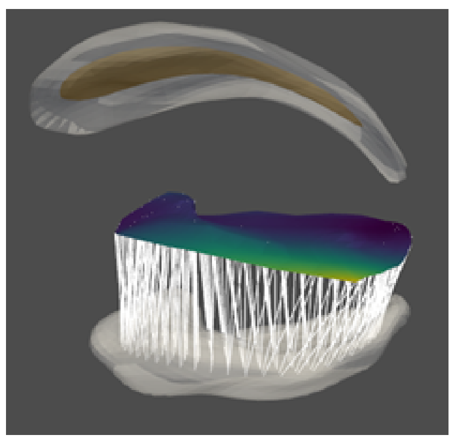


Fig. 1.26 Above: a caudate nucleus with its skeleton. Below: that person's hippocampus. Middle: Liu's linking surface, with links to the hippocampus shown.

1.4.5 Generalized cylinders

One type of anatomic object of interest in Steve's group was the generalized cylinder, which could model the many sorts of tubes in the human body. The generalized cylinder was defined as a possibly curving axis with shapes on cross-sectional planes within some family that had parameters that could vary as the cross-sections passed along the axis. One of the concerns was whether the cross-sectional planes could intersect within the object. Jim studied this question and generated what he called the "relative curvature criterion". This criterion related the axis curvature and the distances from the axis to the shape on the corresponding cross-section. When satisfied, the desired non-intersection of adjacent planes was guaranteed.

Under advising from Steve and Jim, Ruibin Ma studied how to reshape, e.g., straighten, the axis of a generalized cylinder while minimally changing its shape properties (Paper 96 and No. 4 in §1.6.2). For example, straightening a colon could allow a display by slitting it and spreading it open. Damon's relative curvature criterion was an important component of Ma's method. See Figure 1.27.

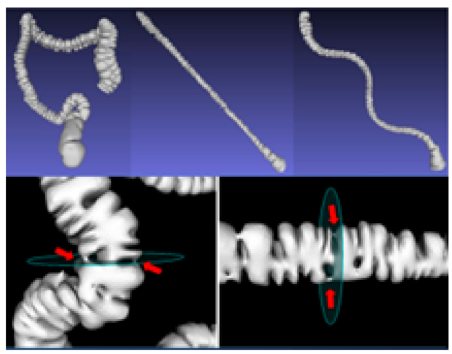


Fig. 1.27 Above: a caudate nucleus with its skeleton. Below: that person's hippocampus. Middle: Liu's linking surface, with links to the hippocampus shown.

Steve was often asked, what was the simplest form of an object which was appropriate for representation by and statistics on an s-rep. Steve called such objects "slabular". Jim provided a definition in terms of swept, non-intersecting cross-sectional planes. Taheri [TPS] has picked up this definition and formed a sort of s-rep that consists of planar s-reps on the cross-sections with the centers of the planar s-reps forming a smooth curve as one passes among the planes, thus viewing the s-rep as a generalized cylinder; see Figure 1.28.

Following Steve's idea that the 1-dimensional skeleton of a 2-dimensional skeleton of a 3-dimensional object would be an important entity, which he called the object's "spine", Taheri called his axis the spine. In his method for fitting the spine and the cross-sections to the target object, he made important use of Damon's relative curvature criterion. The property of Taheri's s-reps that all the spokes in the skeleton within the cross-section associated with any spine point are coplanar has promise for producing better performing statistics due to improved geometric correspondences. See Figure 1.29.

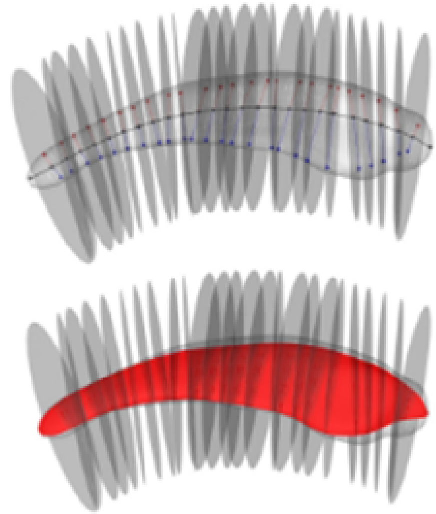


Fig. 1.28 Example of skeletal model of Taheri for a caudate nucleus. Top: discrete slicing planes through spine points (the center curve). Bottom: the slicing planes shown on the skeletal surface (red).

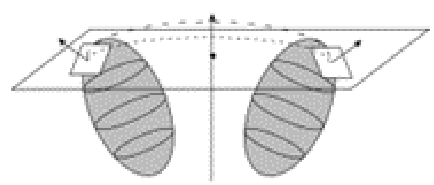


Fig. 1.29 When one of these two ellipsoids should be transformed into the other, the transformation should be a rotation. This was produced as the minimally expensive transformation according to the Lorentzian metric. (Paper 104.)

Separate from the work on skeletal modeling, one of Steve's dissertation advisees [LFP] was concerned with how to produce an interpolating flow between different poses of the same object. The methods that were then available had the unfortunate property of varying the scale of the object between two poses where the scale was the same. Jim developed and published a paper in which he showed that using the Lorentzian metric to produce the flow solved the problem.

1.4.6 Principal component analysis

One of the problems of statistics on shape is producing a mean from a set of objects. More generally, one wanted a set of principal components, such as principal component analysis produces. But the shape representations, s-reps, models including boundary normals, even aligned boundary point models live in an abstract shape space that is curved; in particular on a polysphere (a Cartesian product of spheres and Euclidean spaces). J.S. Marron's dissertation advisee, S. Jung had devised a method of producing a PCA-like analysis of data on a sphere that worked from the dimension of the sphere, iteratively 1 dimension less at a stage, by fitting a best fitting subsphere. He called this method "Principal Nested Spheres" (PNS). When dimension 0 was reached, this produced a sort of mean. He showed that this approach, applied in Euclidean spaces, which he called "backwards PCA", produced the same mean (the Fréchet mean) and principal components as ordinary PCA, but that when it was applied to data on the sphere, the principal spheres and the sphere of dimension 0, i.e., the backwards mean, had the desirable property of staying within the data, unlike the Fréchet mean. A collaboration on this backwards analysis approach between Steve Marron with Jim began at a meeting in southern Denmark, where a presentation by Steve raised the question of why the backwards approach to PCA seemed to be broadly generalizable. Jim made the fundamental observation that the issue became clear if the conventional view of PCA as a series of linear approximations was replaced by a view of PCA-like methods as a system of constraints that could be sequentially removed in a natural way.

1.4.7 Conclusion

Not only did Jim make many wonderful contributions to medical image computing, the mathematics he produced stimulated by these objectives were serious contributions to singularity theory and geometry. And his influence has not stopped. Today's research, beyond his death, is still ongoing. Examples are Taheri's swept plane skeletal models, Pizer's geometric features by s-rep-maintaining diffeomorphisms from an ellipsoid, and a variety of web-resident methods for shape analysis in the SlicerSALT toolkit [Vi] in the development of which Damon's math formed an important role.

1.5 Jim as a colleague

This survey of Jim's mathematical work would not be complete without some mention of his extraordinary generosity as a colleague. For Jim, mathematical research was a collaborative enterprise and not a competitive game. He was always happy to share his ideas and include others in his projects. He was always very quick to praise and give credit to the work of other researchers, and was always encouraging and helpful, as anyone who has worked with him can testify. Meeting him at a conference was one of its high points; he contributed an atmosphere of enthusiasm, enjoyment and scientific curiosity. There are few like him.

1.6 Jim Damon's papers

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1.6.1 Doctoral dissertations in mathematics with Jim Damon as advisor

1. Jason Miller, *Relative Critical Sets in* $\mathbb{R}^n$ *and Applications to Image Analysis*, 1998.
2. Robert Keller, *Generic Transitions of Relative Critical Sets in Parametrized Families with Applications to Image Analysis*, 1999.
3. Brian Pike, *Singular Milnor Numbers of Non-Isolated Matrix Singularities*, 2010.
4. Joel Pereira, *On the Cohen-Macaulay Property of Quotients of Conical Algebras by Monomial Ideals*, 2012.
5. Ellen Gasparovic, *The Blum Medial Linking Structure for Multi-Region Analysis*, 2012. (Steve Pizer was also on the advisory committee.)

1.6.2 Computer science dissertations co-advised by Jim Damon

1. P.Thomas Fletcher, *Statistical Variability in Nonlinear Spaces: Application to Shape Analysis and DT-MRI.* Advisor: Prof. Stephen Pizer, 2004.
2. Qiong Han, *Proper Shape Representation of Single Figure and Multi-Figure Anatomical Objects.* Advisor: Prof. Stephen Pizer, 2008.
3. Ruibin Ma, *Computer-aided Visualization of Colonoscopy.* Advisor: Prof. Stephen Pizer and Jan-Michael Frahm, 2020.
4. Liyun Tu, *Skeletal Shape Correpondence through Entropy.* Advisor: Prof. Stephen Pizer, Dissertation is at Chongqing University, 2018.
5. Jared Vicory, *Shape Deformation Statistics and Regional Texture-based Appearance Models for Segmentation.* Advisor: Prof. Stephen Pizer, 2016.

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