Medial Linking: Generalization of the Blum Medial Axis to Multiple Objects

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Joint work with Jim Damon

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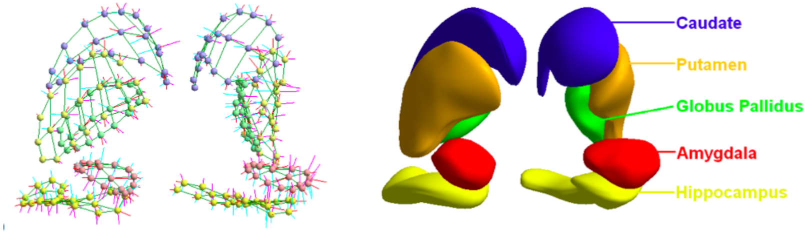
Multi-object complexes



- Objective: Extend medial analysis to collections of objects

1. Classification of generic medial linking structure
   - Components of linking structure
   - Classification results ($n = 2$)
   - Classification results ($n = 3$)
2. Applications to multi-object shape analysis
   - Computations on the linking structure
     - Measures of closeness
     - Measures of significance
   - "Tiered" graph structure and hierarchy
3. Future work and open questions

Maxwell set definition of Blum medial axis

Definition

Suppose $\mathcal{B}$ bounds a region Ω, and let ρ denote the family of distance to the boundary functions, i.e.,

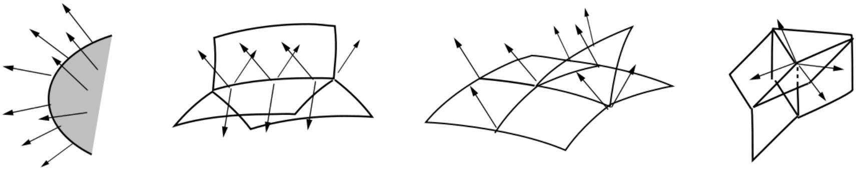
$$\rho : \mathcal{B} \times \mathbb{R}^n \to \mathbb{R},\ (x, u) \mapsto ||x - u||^2.$$

The **Blum medial axis** of Ω is the Maxwell set of ρ:

$$\{u \in \mathbb{R}^n : \exists\, x_1 \neq x_2 \in \mathcal{B} \text{ with } \rho(x_1, u) = \rho(x_2, u) \text{ an absolute min.}\}$$

Generic local forms of medial axis ($n = 2, 3$)

- $\mathcal{A}_1^2$: smooth curve/sheet
- $\mathcal{A}_3$: edge point/curve, $\mathcal{A}_1^3$: branch point/curve
- ($n = 3$ only) $\mathcal{A}_1\mathcal{A}_3$: fin point, $\mathcal{A}_1^4$: 6–junction point



a) edge b) Y-branching c) fin creation point d) "6-junction"

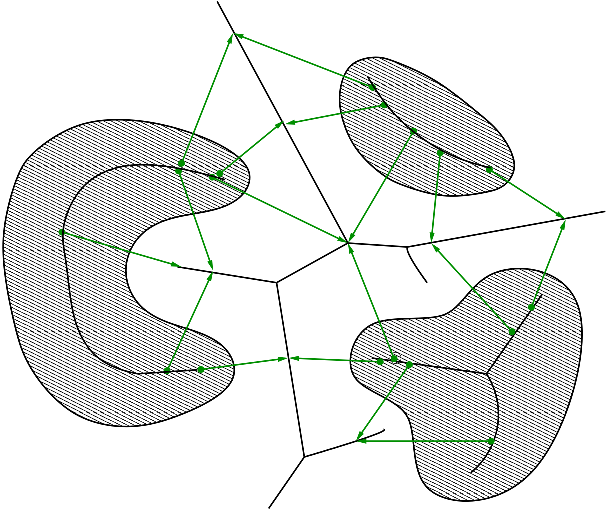
Definition of medial linking structure

Definition

A **medial linking structure** associated to a multi-object complex $\{\Omega_1, ..., \Omega_n\}$ consists of the following components:

1. the collection of Blum medial axes M_i and associated radial vector fields $U_i = r_i \mathbf{u}_i$;
2. a collection of multivalued *linking functions* $\ell_i : M_i \to \mathbb{R}^+$;
3. a collection of multivalued *linking vector fields* $L_i = \ell_i \mathbf{u}_i$, one for each M_i; and
4. labeled refinements $\mathcal{S}_i$ of the Whitney stratifications of the medial axes.

2D linking example



Linking functions and vector fields

Definition

For each M_i, we define a **linking function** $\ell_i : M_i \to \mathbb{R}^+$ which is characterized by the following properties:

1. ℓ_i is continuous;
2. ℓ_i is smooth on every stratum of $\mathcal{S}_i$; and
3. $\ell_i(x) \geq r_i(x)$ for all $x \in M_i$.

Definition

Given a point $x \in M_i$ and a choice of unit radial vector $\mathbf{u}_i(x)$, define the **linking vector** at x as $L_i(x) = \ell_i(x)\mathbf{u}_i(x)$. The collection of all such L_i is called the **linking vector field** on M_i.

Definition of linking

Definition

Two points $x \in M_i$ and $y \in M_j$ are said to be **linked** if for some choice of linking vectors $L_i(x)$ and $L_j(y)$,

$$x + L_i(x) = y + L_j(y).$$

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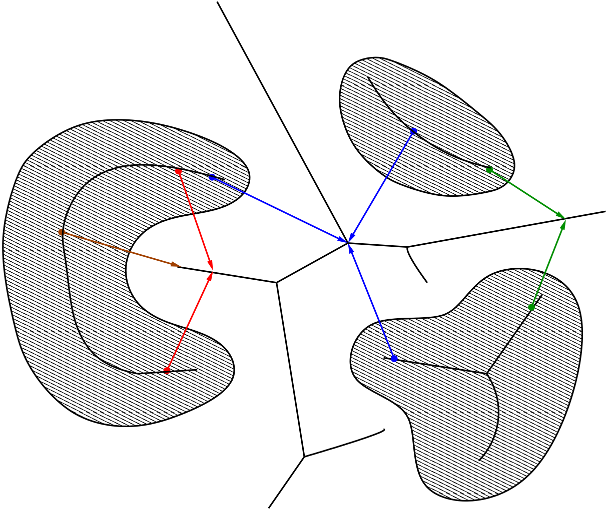
$$x + L_i(x) = y + L_j(y).$$

- In the Blum case, if $x \in M_i$ and $y \in M_j$ are linked, the linking functions satisfy

$$\ell_i(x) - r_i(x) = \ell_j(y) - r_j(y).$$

2D linking example



Use of the medial axis "double"

- Want to simultaneously consider "both sides" of M

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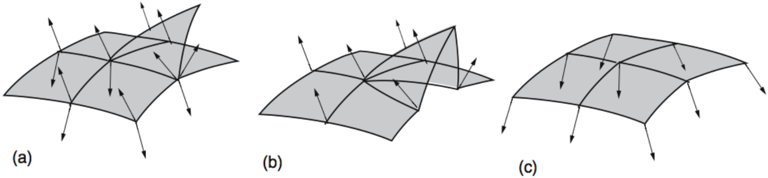
- Want to simultaneously consider "both sides" of M
- $\tilde{M} = \{(x, U') \in M \times \mathbb{R}^{n+1} \mid U' \text{ is a value of } U \text{ at } x\}$



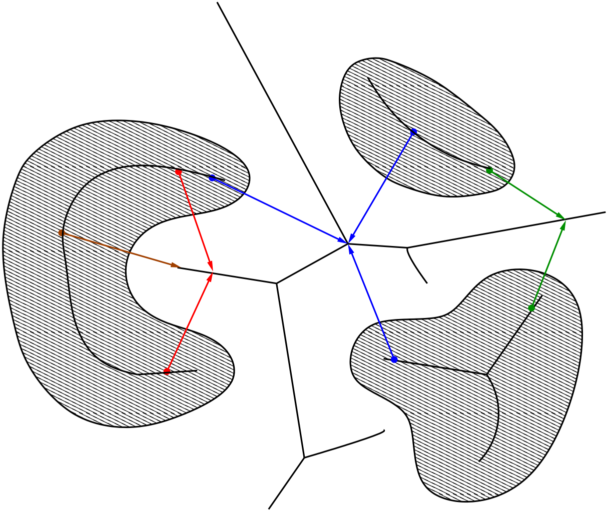
Labeled refinements to Whitney stratifications of $\tilde{M}_i$'s

Category 1: Singular points on internal medial axes linked to points on other internal medial axes at smooth point of linking medial axis.
Category 2: Smooth points on internal medial axes linked at singular point of linking medial axis.
Category 1/2: Singular points on internal medial axes linked at singular point of linking medial axis.

Later:

Category 3: Points on internal medial axes linked at intersection of enclosing region with linking medial axis.

2D linking example



Generic linking between distinct medial axes ($n = 2$)

- Collection of disjoint regions bounded by smooth curves

List of normal forms for linking structure

- 3 possibilities at a smooth curve of linking medial axis; and
- 1 possibility at a branch point of linking medial axis.

Generic linking between distinct medial axes ($n = 2$)

- Collection of disjoint regions bounded by smooth curves

List of 4 normal forms for linking structure

- 3 possibilities at a smooth curve of linking medial axis:

$$(\mathcal{A}_1^2 : \mathcal{A}_1^2, \mathcal{A}_1^2),\ (\mathcal{A}_1^2 : \mathcal{A}_1^3, \mathcal{A}_1^2),\ (\mathcal{A}_1^2 : \mathcal{A}_3, \mathcal{A}_1^2)$$

- 1 possibility at a branch point of linking medial axis:

$$(\mathcal{A}_1^3 : \mathcal{A}_1^2, \mathcal{A}_1^2, \mathcal{A}_1^2)$$

Generic self-linking ($n = 2$)

List of normal forms for self-linking

- 3 possibilities at a smooth curve of linking medial axis
- 1 possibility at a branch point of linking medial axis
- **1 possibility at an edge point of linking medial axis:**

$$(\mathcal{A}_3 : \mathcal{A}_1^2)$$

Generic self-linking ($n = 2$)

List of normal forms for self-linking

- 3 possibilities at a smooth curve of linking medial axis;
- 1 possibility at a branch point of linking medial axis; and
- 1 possibility at an edge point of linking medial axis.

- Brings total number of normal forms for linking structure to 5

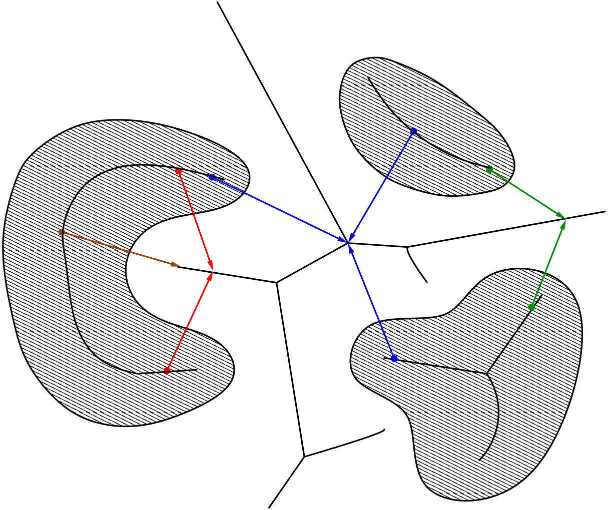
Generic self-linking ($n = 2$)

List of normal forms for self-linking

- 3 possibilities at a smooth curve of linking medial axis;
- 1 possibility at a branch point of linking medial axis; and
- 1 possibility at an edge point of linking medial axis.

- Brings total number of normal forms for linking structure to 5
- Refined stratification just adds points

2D Generic Linking



Generic linking between distinct medial axes ($n = 3$)

- Collection of disjoint regions bounded by smooth surfaces

List of 13 normal forms for linking structure

- 8 possibilities at a smooth sheet of linking medial axis;
- 3 possibilities at a branch curve of linking medial axis; and
- 2 possibilities at a zero-dim'l stratum of linking medial axis.

Generic linking between distinct medial axes ($n = 3$)

- Collection of disjoint regions bounded by smooth surfaces

List of 13 normal forms for linking structure

- **8 possibilities at a smooth sheet of linking medial axis:**
  - Two points on two smooth sheets
  - Point on smooth sheet, point on 1D stratum
  - Point on smooth sheet, point on 0D stratum
  - Two points of transversely intersecting 1D strata
- 3 possibilities at a branch curve of linking medial axis; and
- 2 possibilities at a zero-dim'l stratum of linking medial axis.

Generic linking between distinct medial axes ($n = 3$)

- Collection of disjoint regions bounded by smooth surfaces

List of 13 normal forms for linking structure

- 8 possibilities at a smooth sheet of linking medial axis;
- **3 possibilities at a branch curve of linking medial axis:**
  - Three points on three smooth sheets
  - Two points on two smooth sheets, one point on 1D stratum
- 2 possibilities at a zero-dim'l stratum of linking medial axis.

Generic linking between distinct medial axes ($n = 3$)

- Collection of disjoint regions bounded by smooth surfaces

List of 13 normal forms for linking structure

- 8 possibilities at a smooth sheet of linking medial axis;
- 3 possibilities at a branch curve of linking medial axis; and
- **2 possibilities at a zero-dim'l stratum of linking medial axis:**
    - Two (or four) points on two (or four) smooth sheets

Generic self-linking ($n = 3$)

List of normal forms for self-linking

- 8 possibilities at a smooth sheet of linking medial axis;
- 3 possibilities at a branch curve of linking medial axis;
- 2 possibilities at a zero-dim'l stratum of linking medial axis; and
- **3 possibilities at an edge curve of linking medial axis.**

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- 8 possibilities at a smooth sheet of linking medial axis;
- 3 possibilities at a branch curve of linking medial axis;
- 2 possibilities at a zero-dim'l stratum of linking medial axis; and
- **3 possibilities at an edge curve of linking medial axis:**
    - Point on a smooth sheet
    - Point on a branch curve
    - Point on another edge curve if surface locally a saddle

Generic self-linking ($n = 3$)

List of normal forms for self-linking

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- Brings total number of normal forms for linking structure to 16

Generic self-linking ($n = 3$)

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- 8 possibilities at a smooth sheet of linking medial axis;
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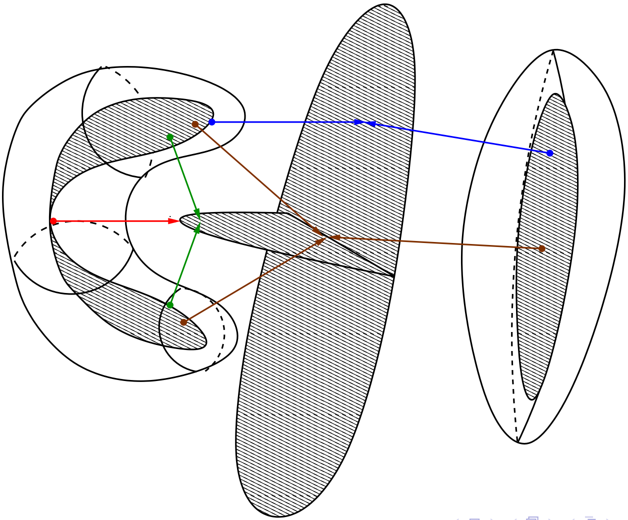
- 3 possibilities at an edge curve of linking medial axis.

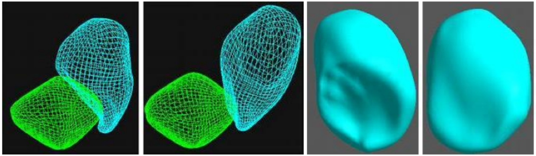
- Brings total number of normal forms for linking structure to 16
- Refined stratification adds curves, points

3D generic linking



Motivating issues in multi-object shape analysis

1. How to capture shape/pose changes from influences of nearby objects?



2. How to determine close or "neighboring" regions?
3. Which objects/regions are most/least significant?
4. How to rigorize correspondence across instances and choice of orderings/scales?

Our approach to these issues

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- Linking structure captures individual, positional/relative geometry
- Candidates for measures of comparison: closeness, significance
- Organization/synthesis of data

Computations on the medial axis

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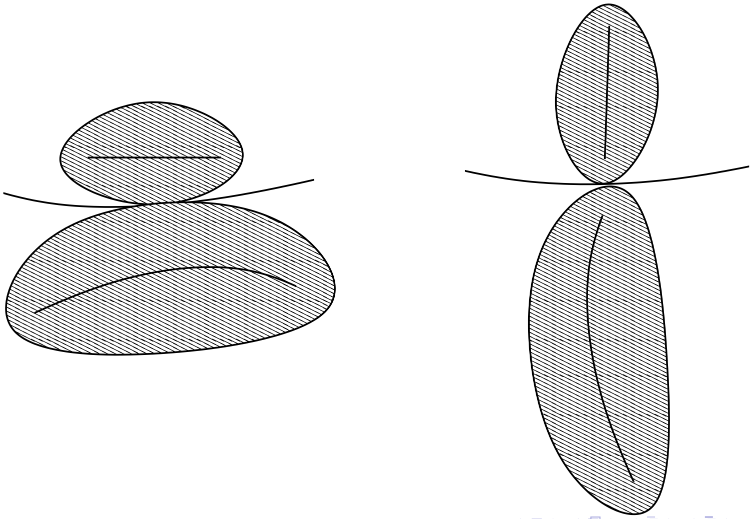
  - Area ($n = 2$) or volume ($n = 3$) given by $\displaystyle\int_{\tilde{M}} \delta \cdot r\, dM$
- Also, can compute integrals over portions of Ω
- Area/volume preserved under perturbations

Example of a closeness measure



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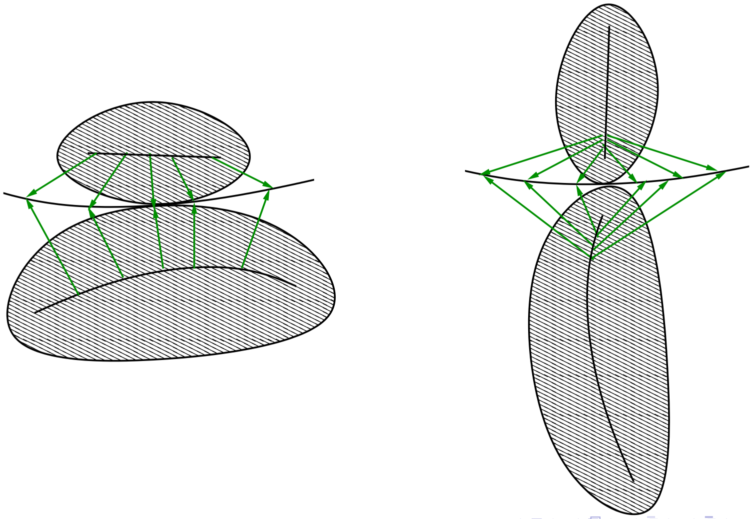

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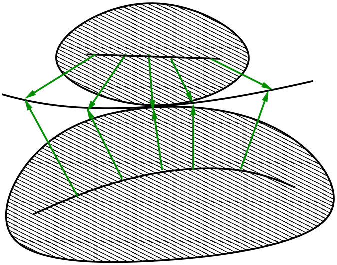
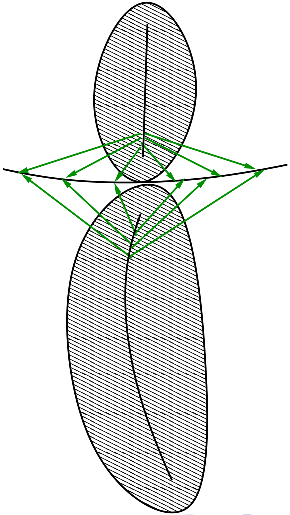
- Possible closeness measure is

$$\boxed{\frac{\mathrm{vol}(\Omega_i \cap R_{i\to j}) + \mathrm{vol}(\Omega_j \cap R_{j\to i})}{\mathrm{vol}(R_{i\to j}) + \mathrm{vol}(R_{j\to i})}}$$

- Gives number between 0 and 1

Example of a closeness measure





Multi-object example to keep in mind

Another candidate for a closeness measure

- Ratio of radial function to linking function

Another candidate for a closeness measure

- Ratio of radial function to linking function
- Yields measure in mathematical sense

$$\boxed{\frac{r}{\ell}\, dV}$$

Global measure of significance

- Volume of object/part of object as measure of significance

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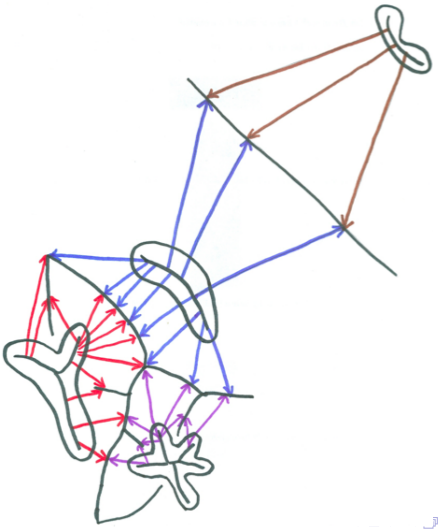
  - $(n = 2)$

$$\boxed{\text{Area}(\Omega) = \int_{\tilde{M}} r \, dM - \frac{1}{2} \int_{\tilde{M}} r^2 \, \kappa_r \, dM}$$

  - $(n = 3)$

$$\boxed{\text{Volume}(\Omega) = \int_{\tilde{M}} r \, dM - \int_{\tilde{M}} r^2 H_{rad} \, dM + \frac{1}{3} \int_{\tilde{M}} r^3 \, K_{rad} \, dM}$$

Candidate for a significance measure



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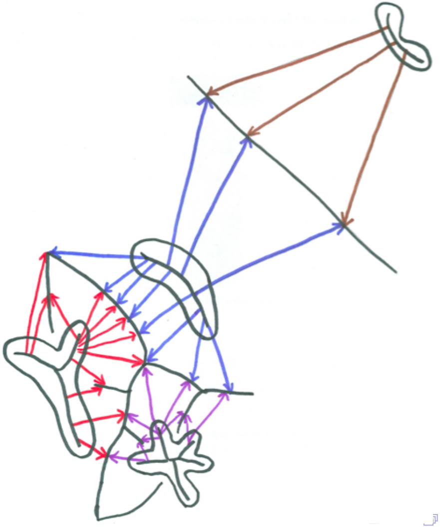
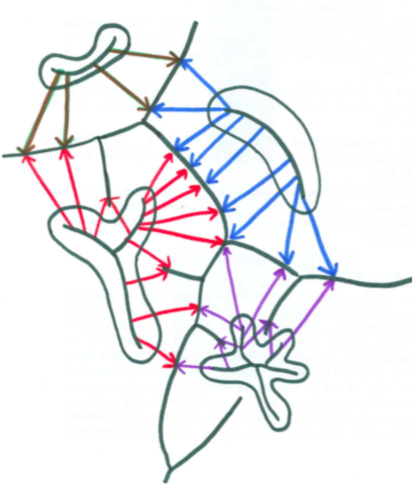
$$\boxed{\frac{\mathrm{vol}(\Omega_i \cap R_i)}{\mathrm{vol}(R_i)}}$$

- Continuous under small generic perturbations of objects

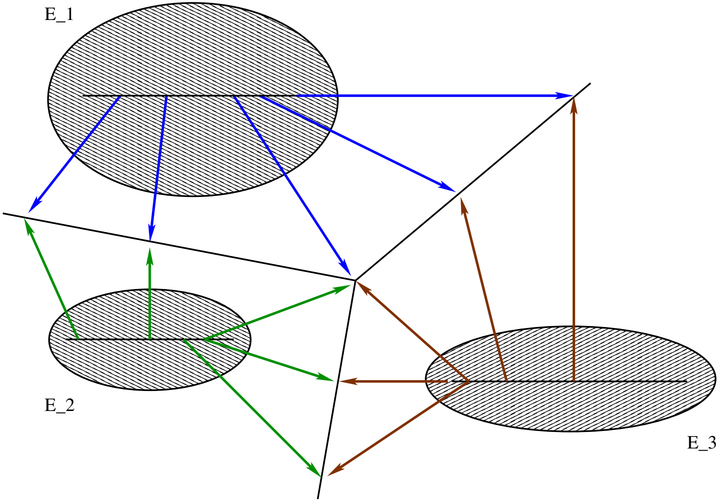
Candidate for a significance measure



Candidate for a significance measure



Simple illustration



Graph-theoretic analysis of comparison measures

- Vertices, edges weighted by significance, closeness measures

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- Vertices, edges weighted by significance, closeness measures
- Weights yield "height functions" on graph – induce different levels to extract orderings among objects
- Stability result for graph structure

In the future...

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- Deformations of objects

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- Presence of tumors, objects within objects

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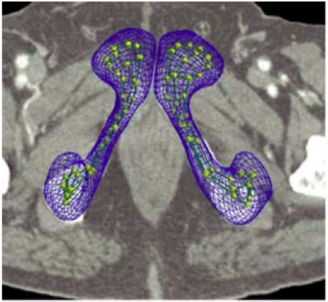
- "X-factor"

More on the "X-factor"

- Segmentation of organs with insufficient boundary intensity



More on the "X-factor"

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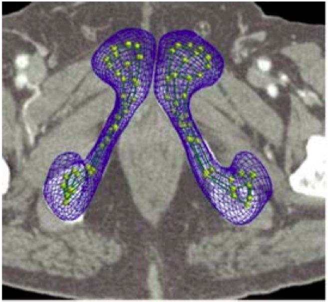

- Additional discrete function on graph